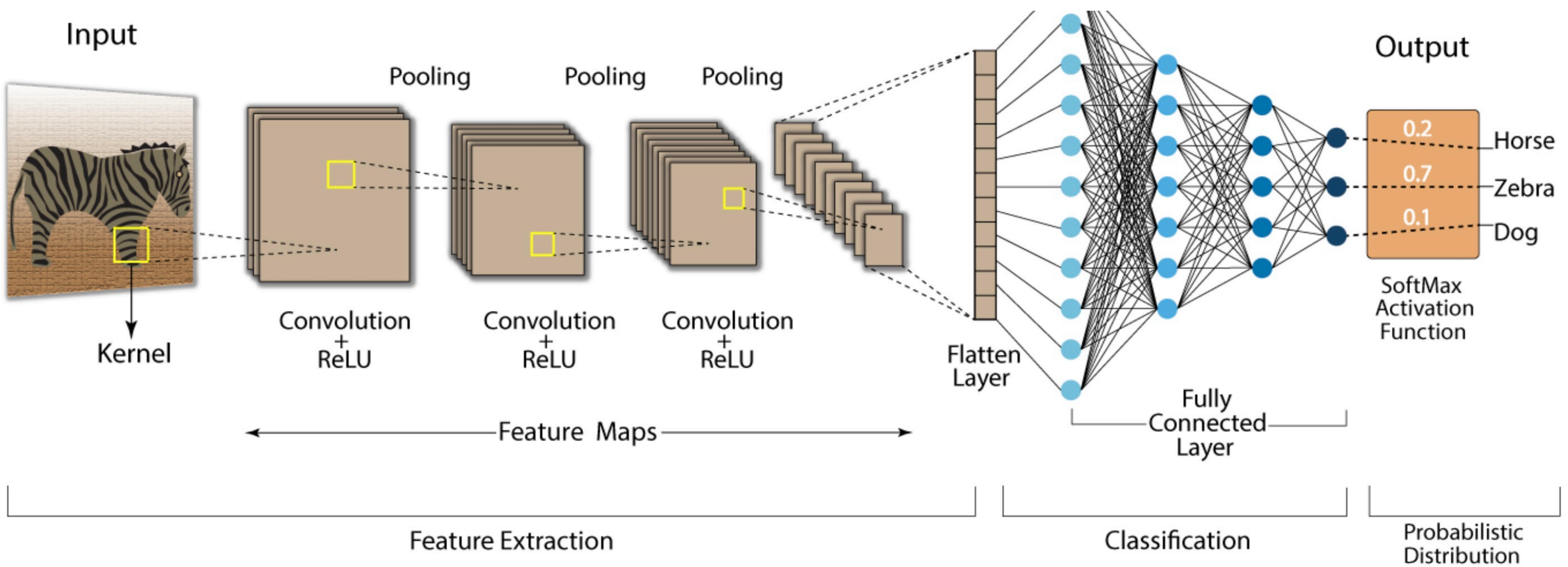
On the Shift Invariance of Max Pooling Feature Maps in CNN

joint work with H. Leterme, K. Alahari et V. Perrier

Kévin Polisano

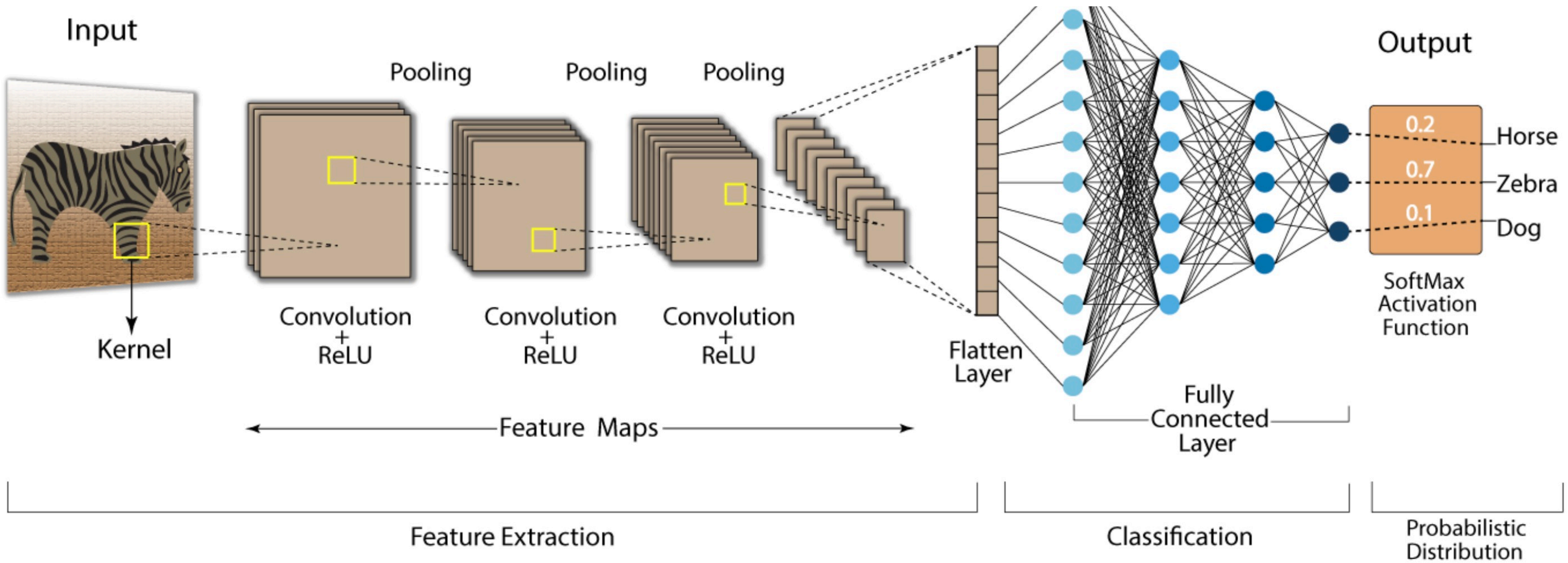
Convolutional neural networks



Source : https://developersbreach.com/convolution-neural-network-deep-learning/

Convolutional neural networks

- Image classification: feature vectors are fed into a linear classifier

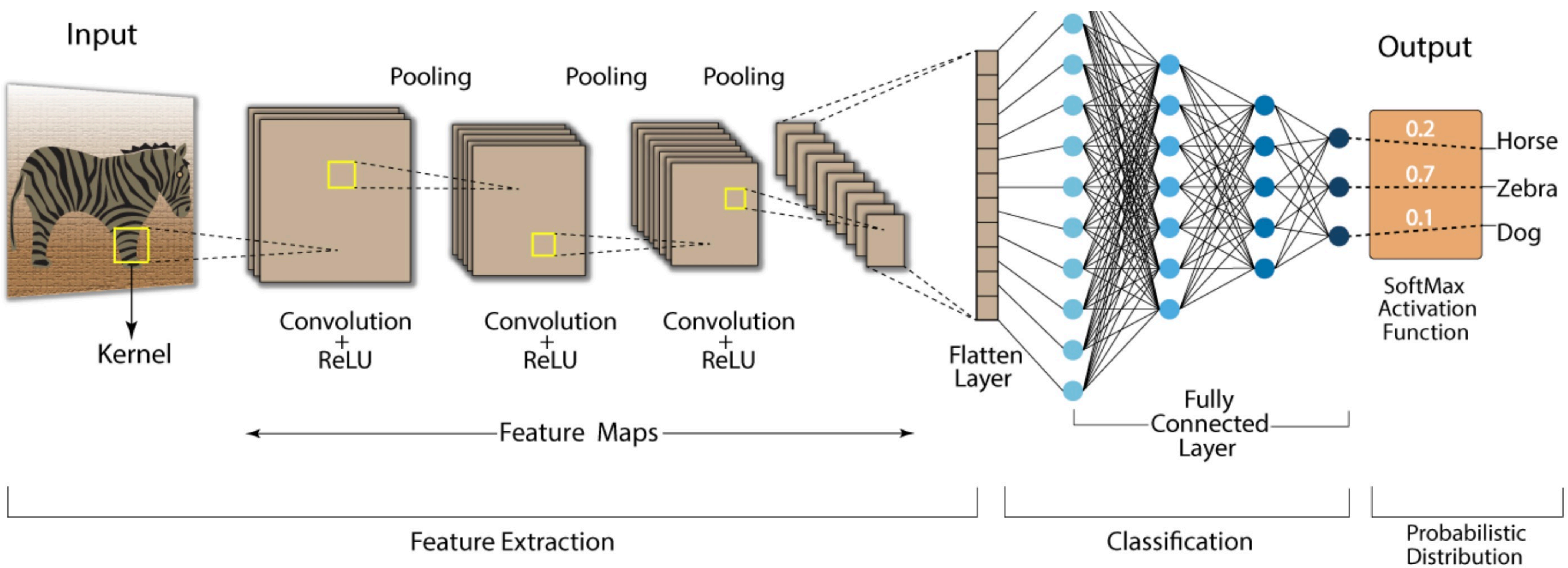


Source : https://developersbreach.com/convolution-neural-network-deep-learning/

Convolutional neural networks

- Image classification: feature vectors are fed into a linear classifier

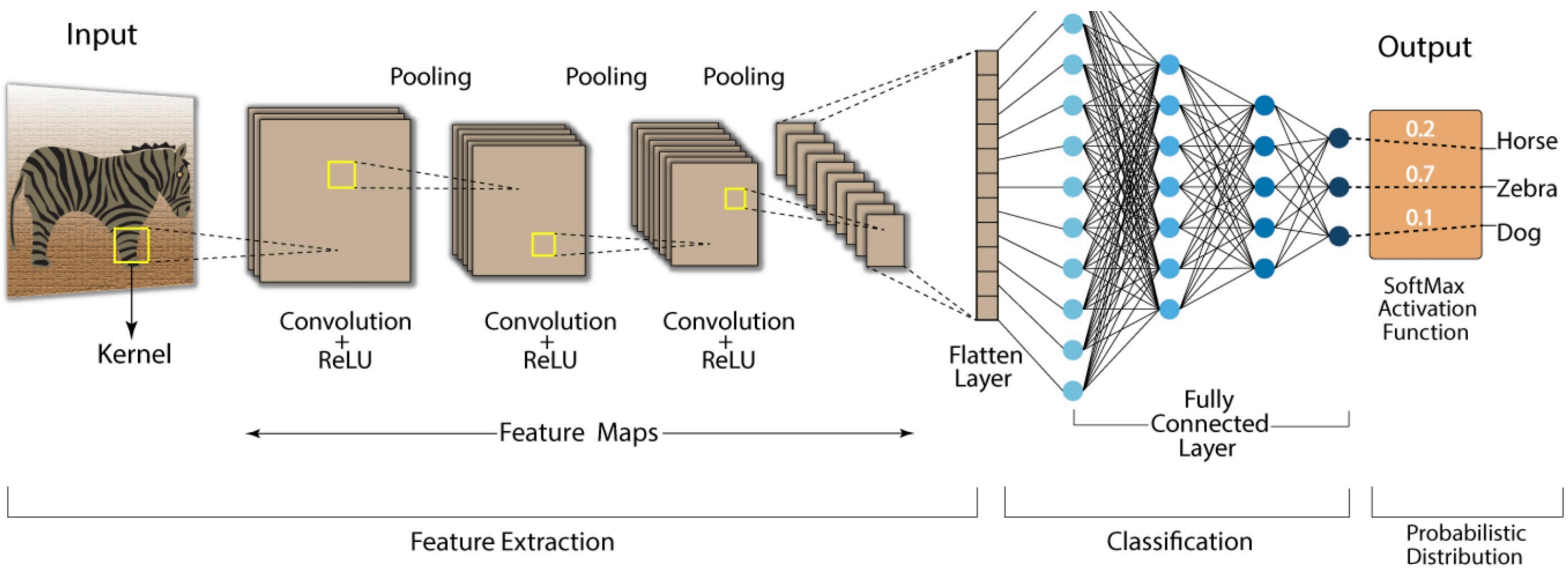
- Desired property of CNN: to remain invariant to small translations



Source : https://developersbreach.com/convolution-neural-network-deep-learning/

Convolutional neural networks

- Image classification: feature vectors are fed into a linear classifier
- Desired property of CNN: to remain invariant to small translations
- Are extracted features maps stable to translations?

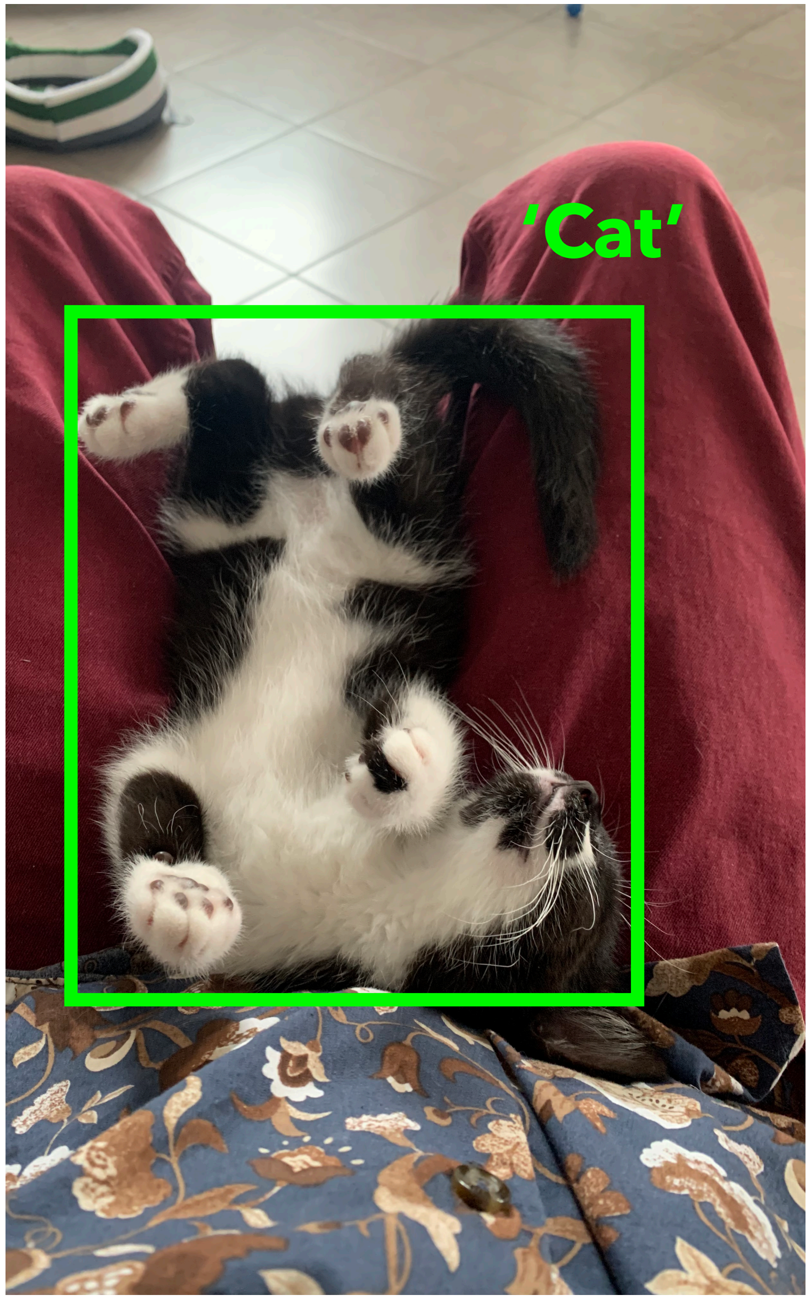


Source : https://developersbreach.com/convolution-neural-network-deep-learning/

Are CNNs shift-invariant?

My cat Ada

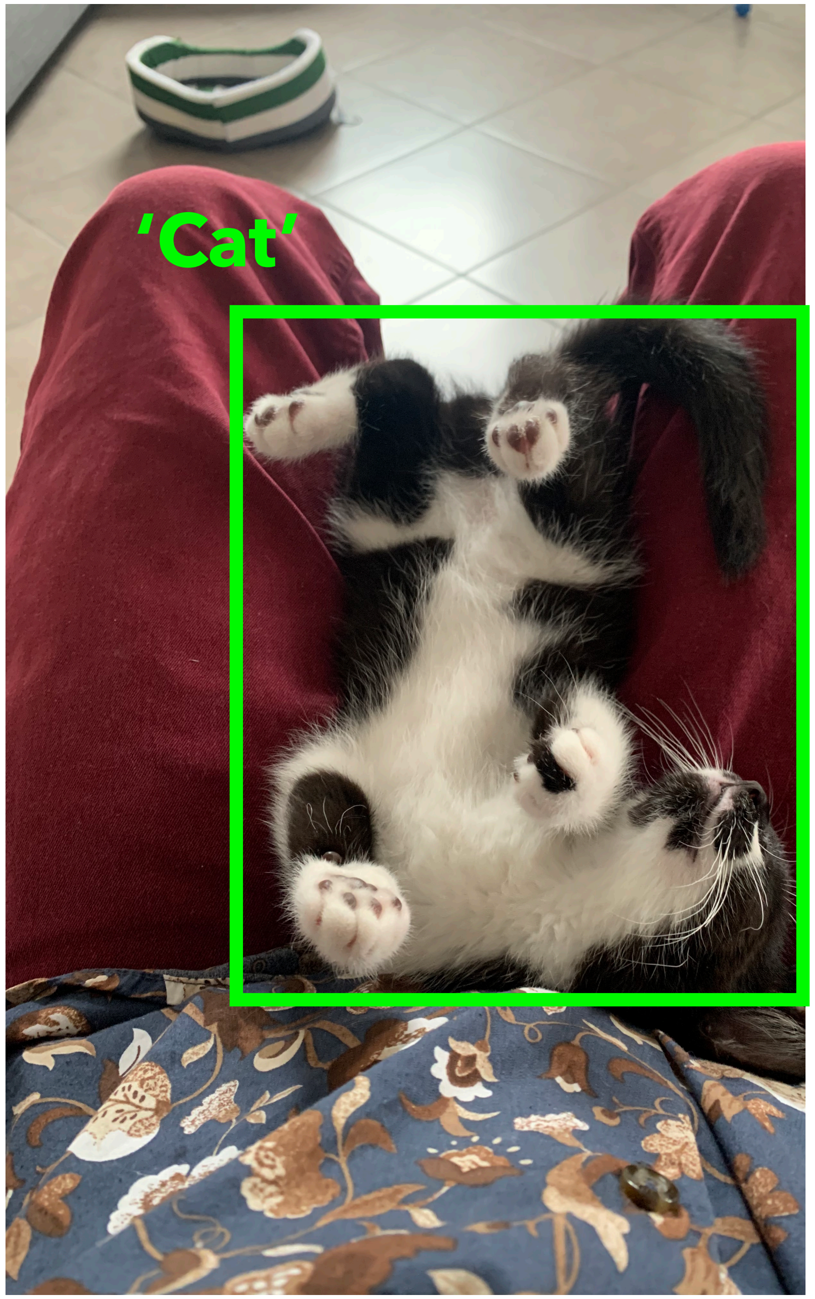
Are CNNs shift-invariant?



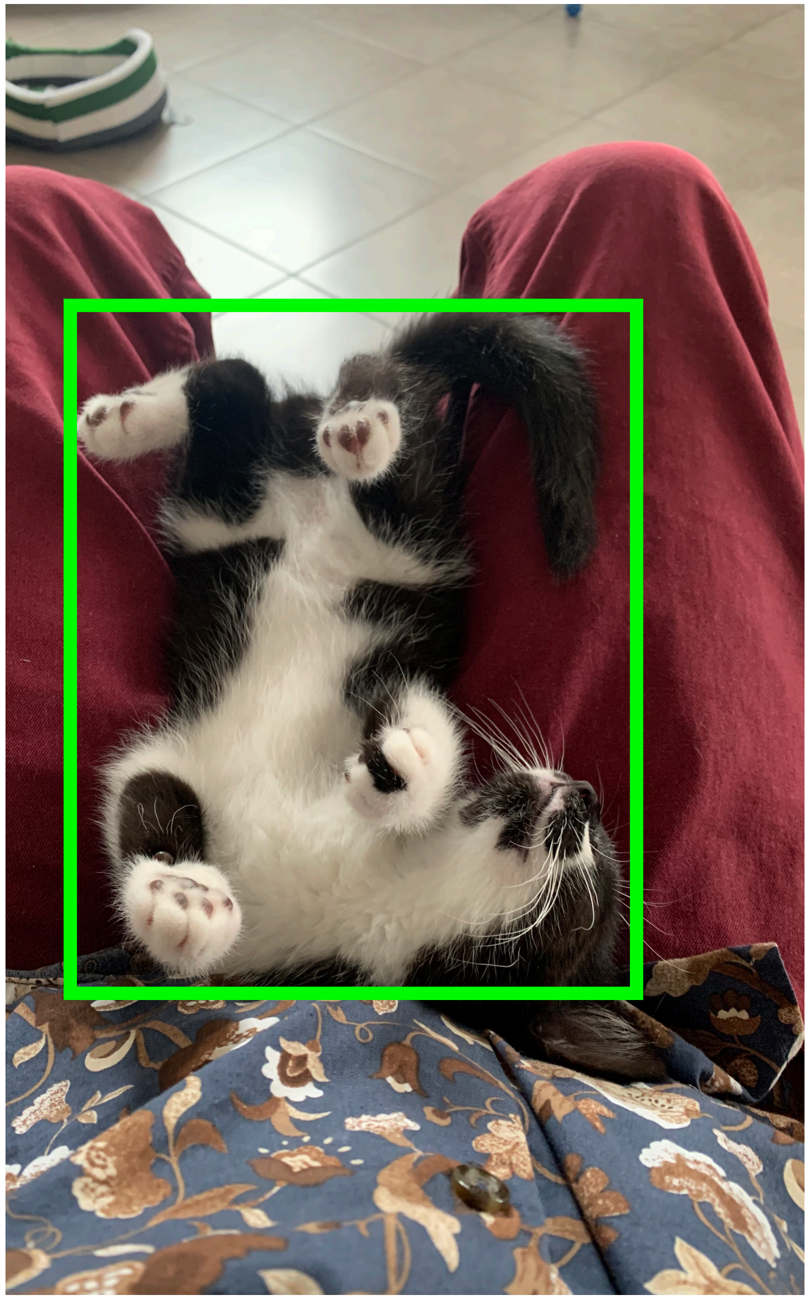
Are CNNs shift-invariant?

Are CNNs shift-invariant?

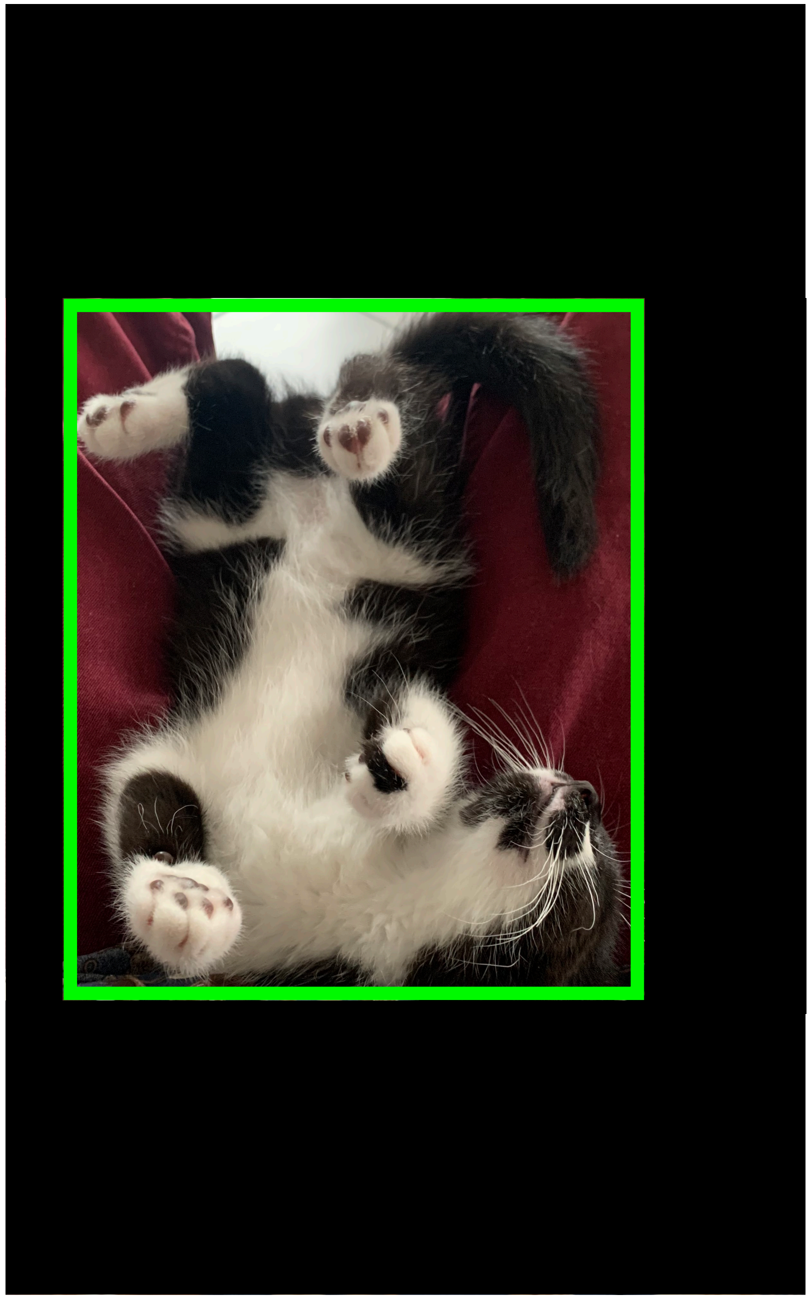
Are CNNs shift-invariant?



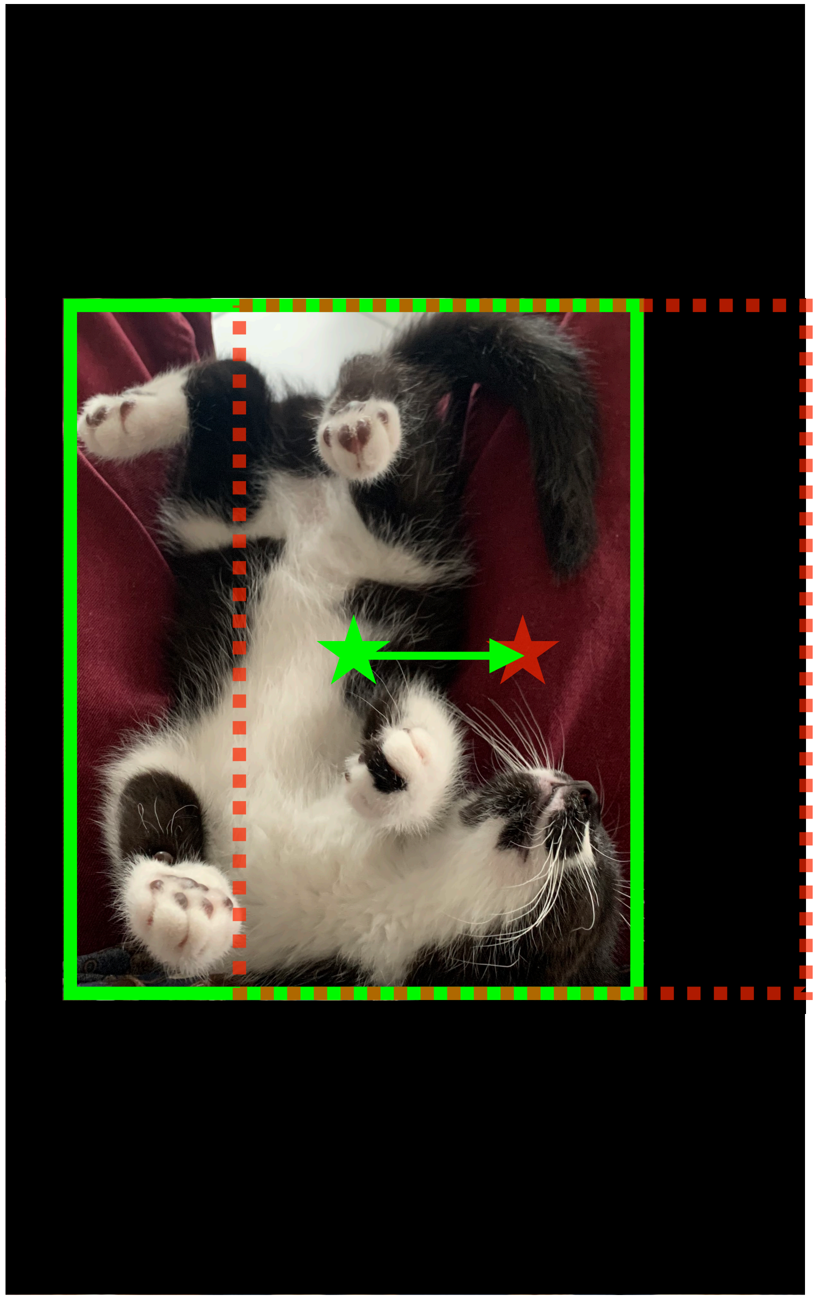
Are CNNs shift-invariant?



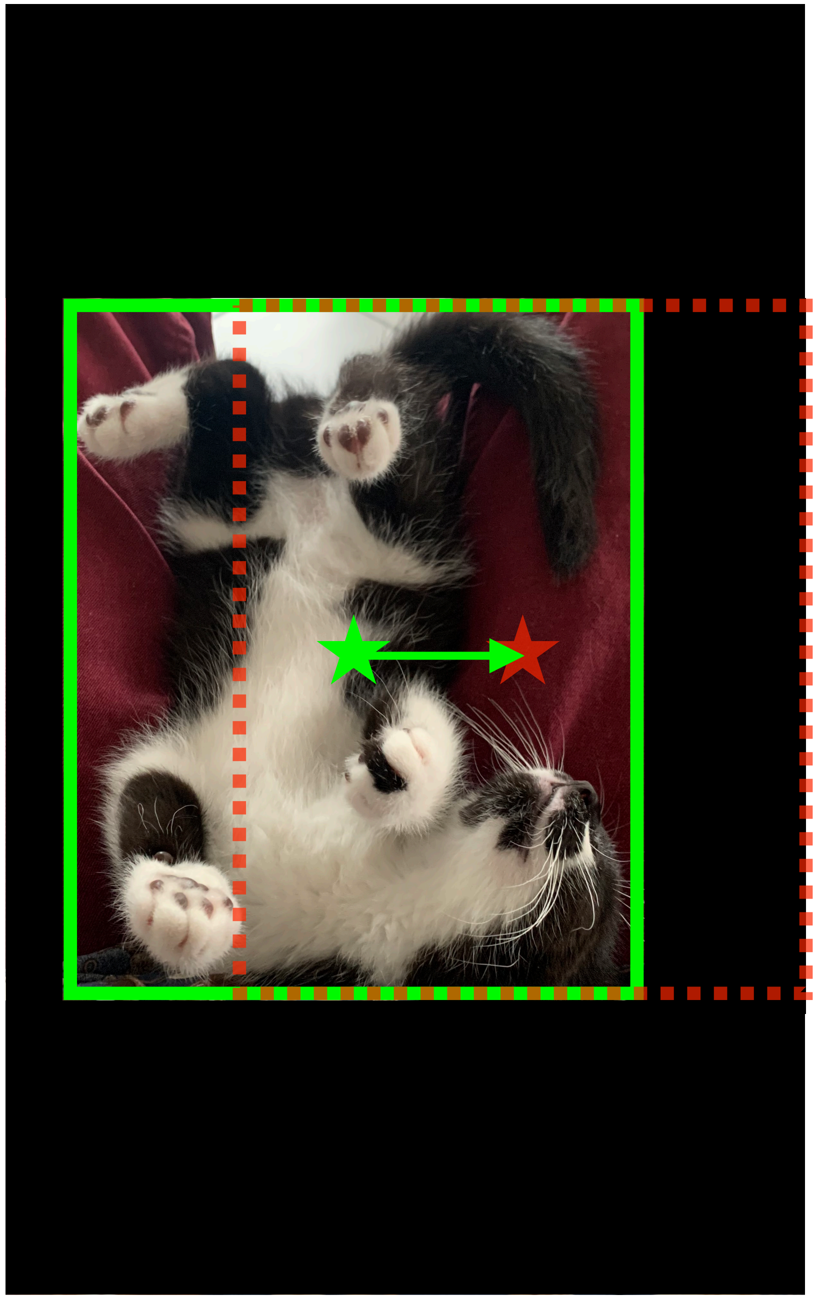
Are CNNs shift-invariant?

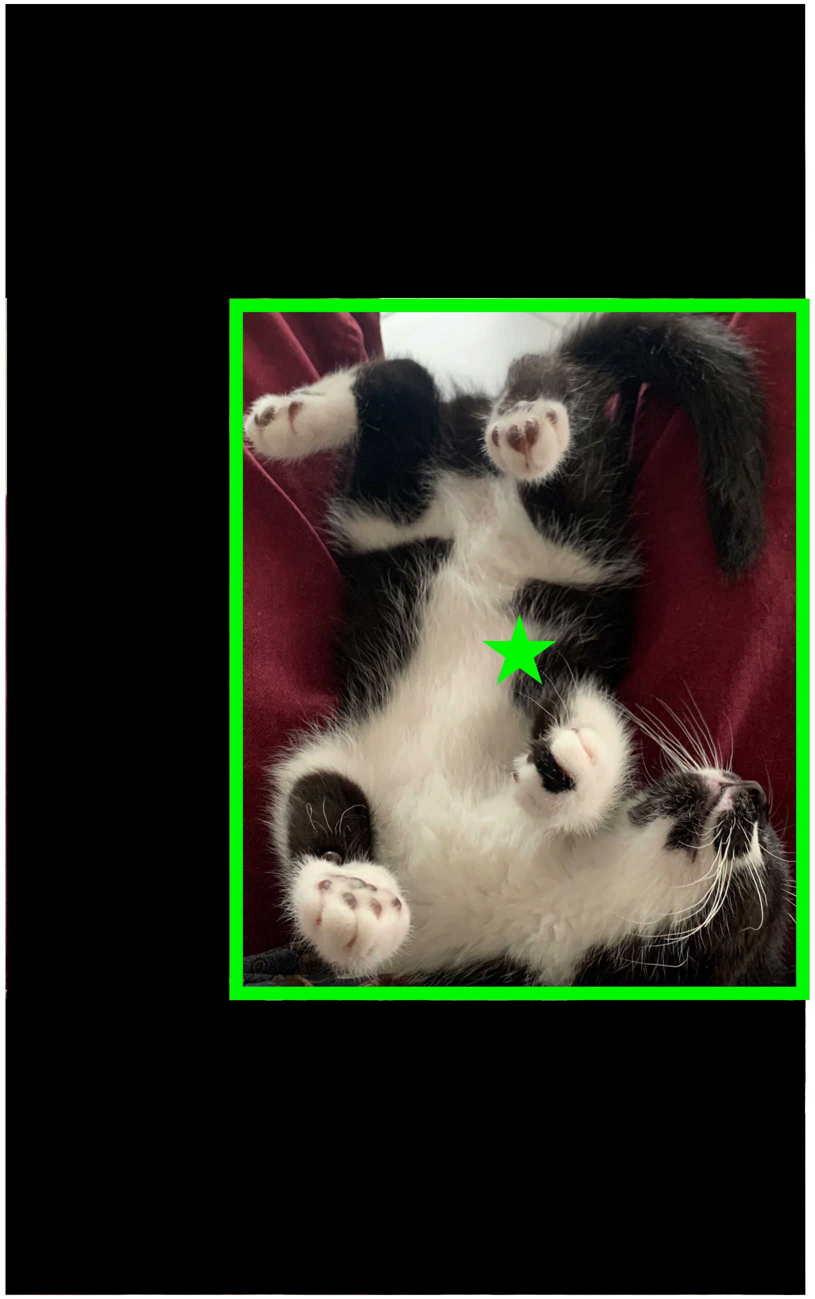


Are CNNs shift-invariant?

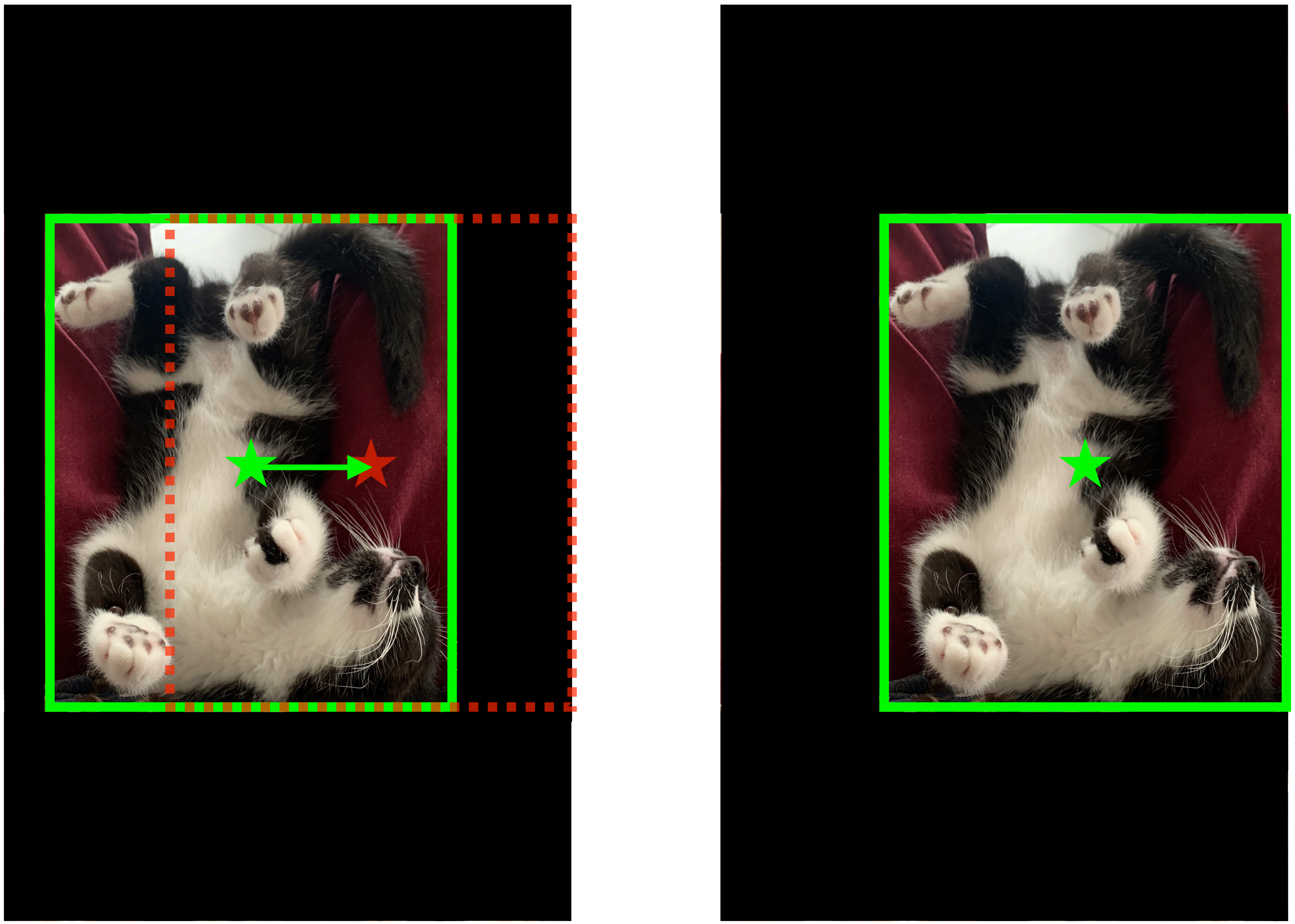


Are CNNs shift-invariant?





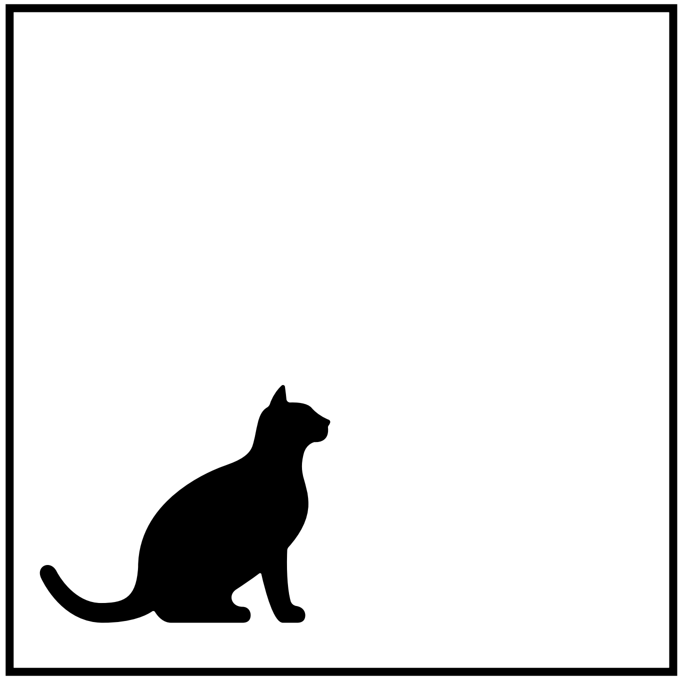
Are CNNs shift-invariant?

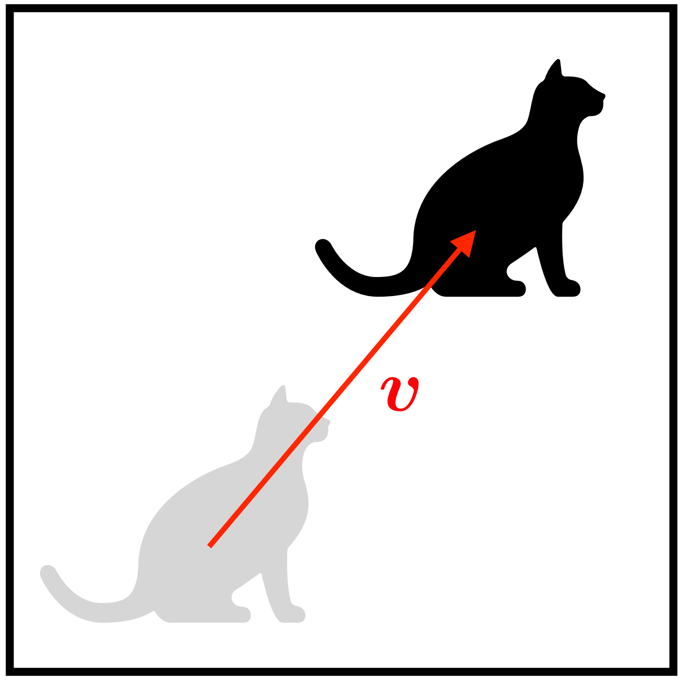


Shift invariance

Input image X

Shifted input $\mathcal{T}_{\boldsymbol{v}}\mathrm{X}$



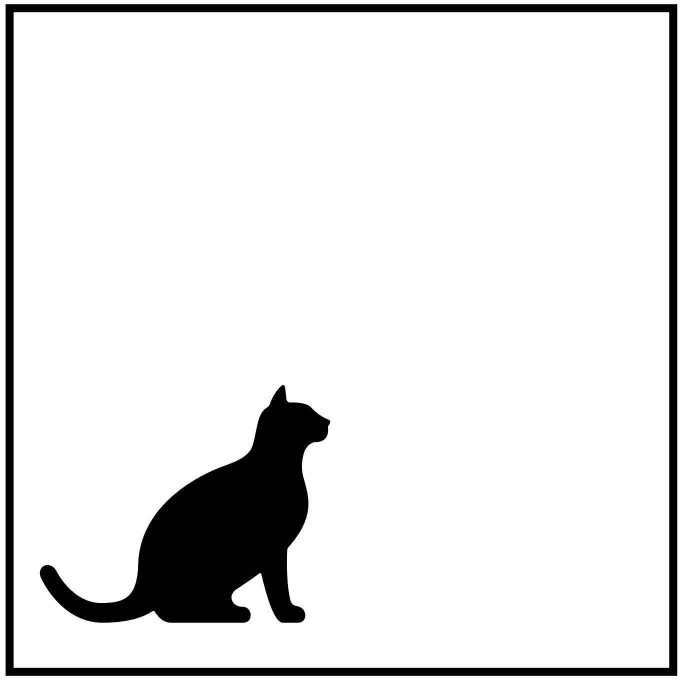


Output $f(\mathrm{X}) = 1$

Output $f(\mathcal{T}_{\boldsymbol{v}}\mathrm{X}) = 1$

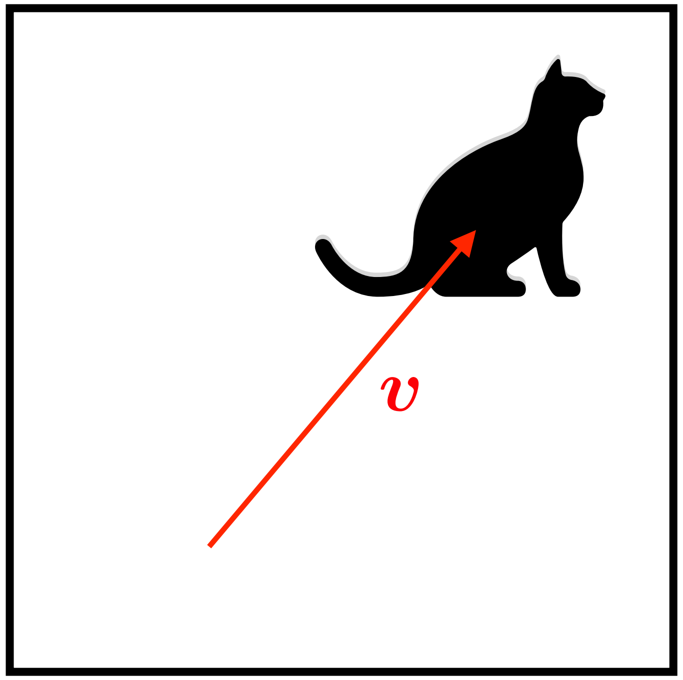
Shift invariance

Input image X



Output $f(\mathrm{X}) = 1$

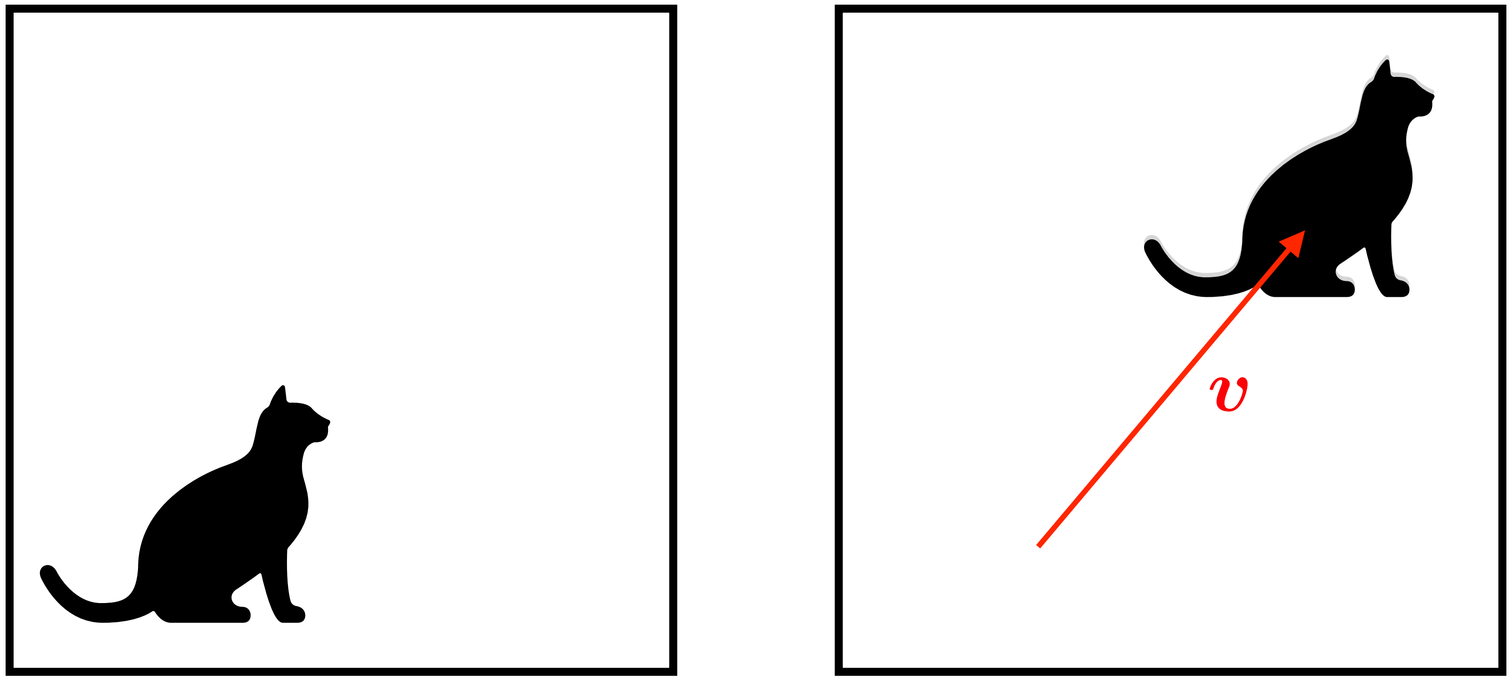
Shifted input $\mathcal{T}_{\boldsymbol{v}}\mathrm{X}$



Output $f(\mathcal{T}_{\boldsymbol{v}}\mathrm{X}) = 1$

Shift invariance

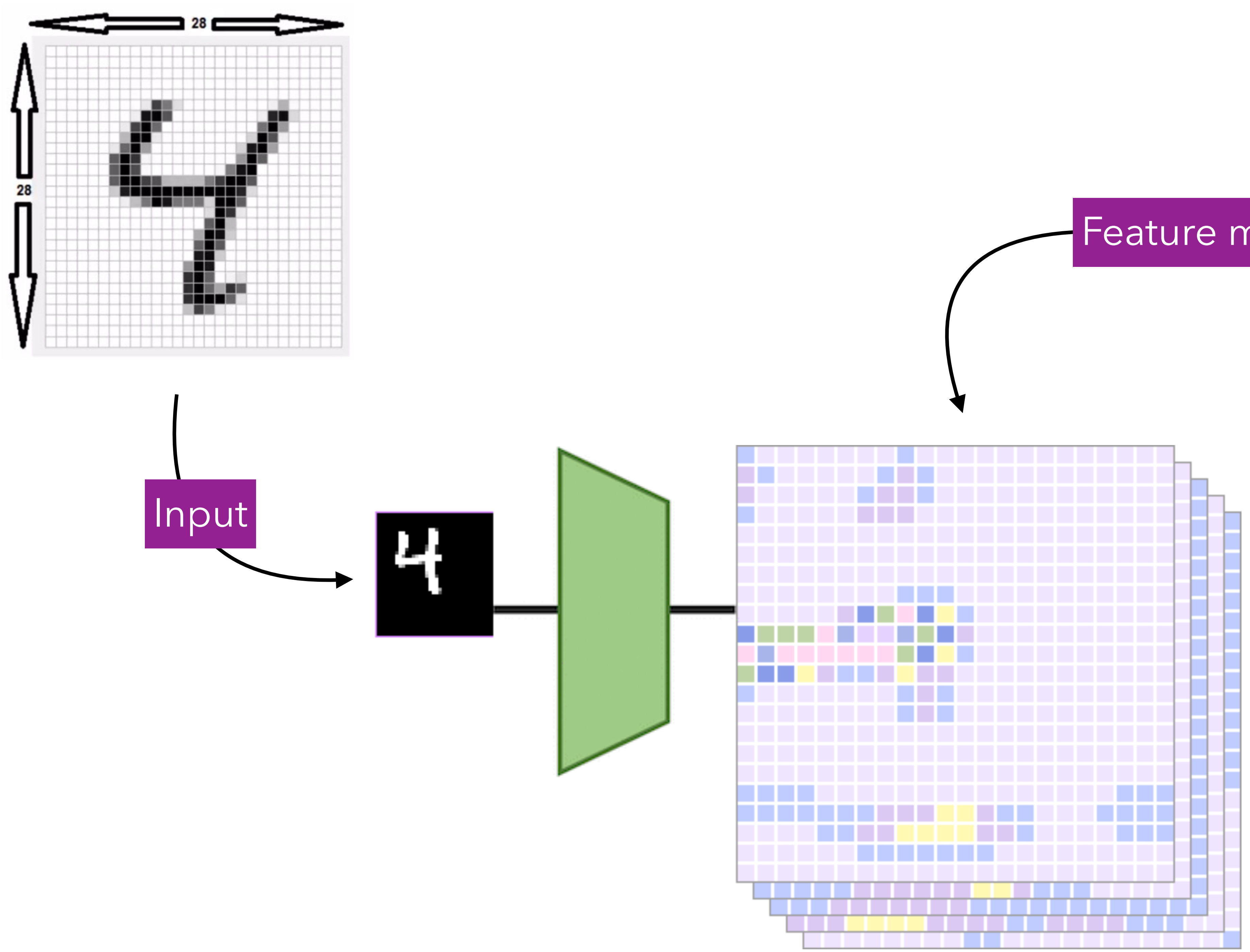
Input image X Shifted input $\mathcal{T}_{\boldsymbol{v}}\mathrm{X}$



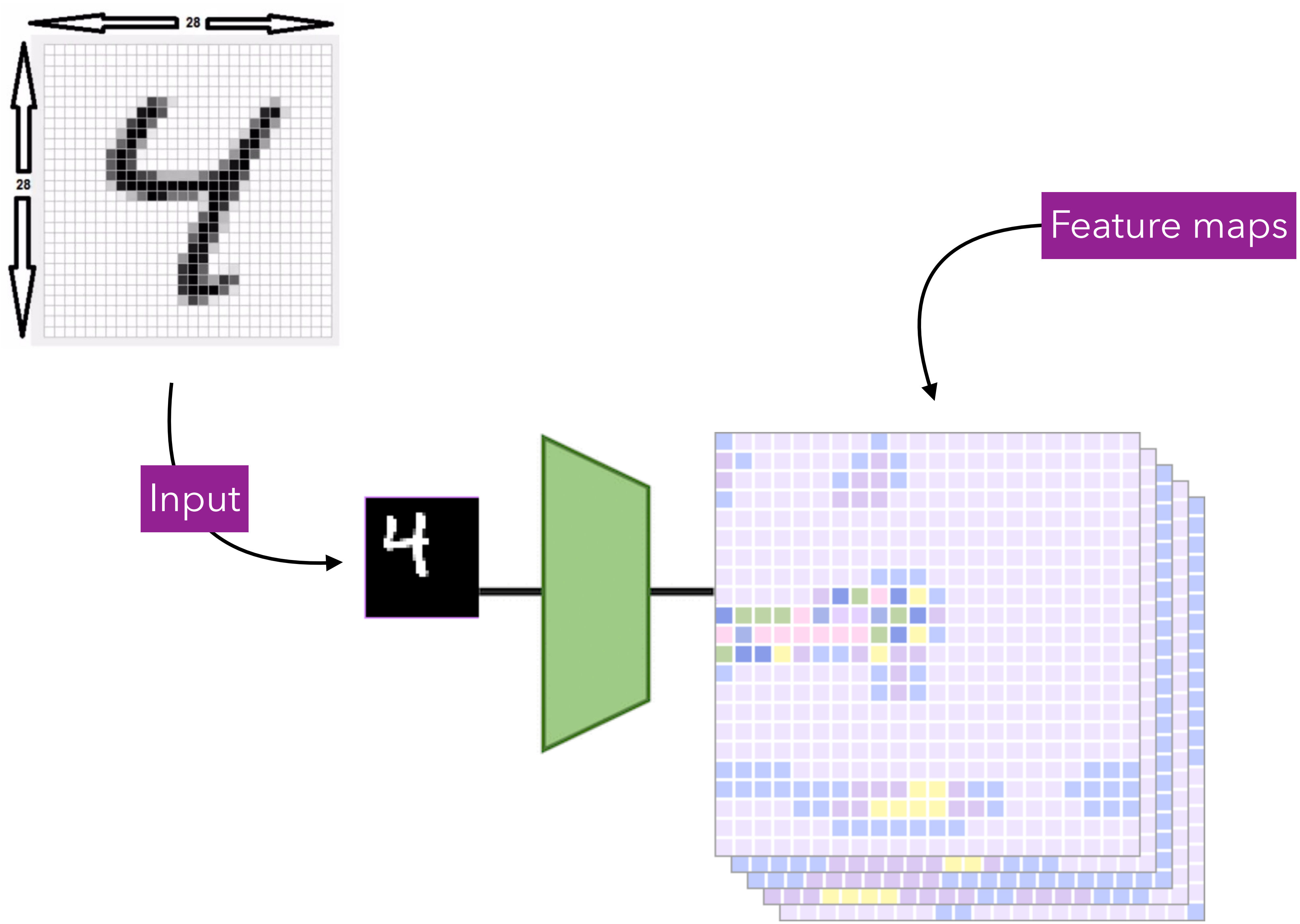
Output $f(\mathrm{X}) = 1$ Output $f(\mathcal{T}_{\boldsymbol{v}}\mathrm{X}) = 1$

Shift invariance ≠ Equivariance

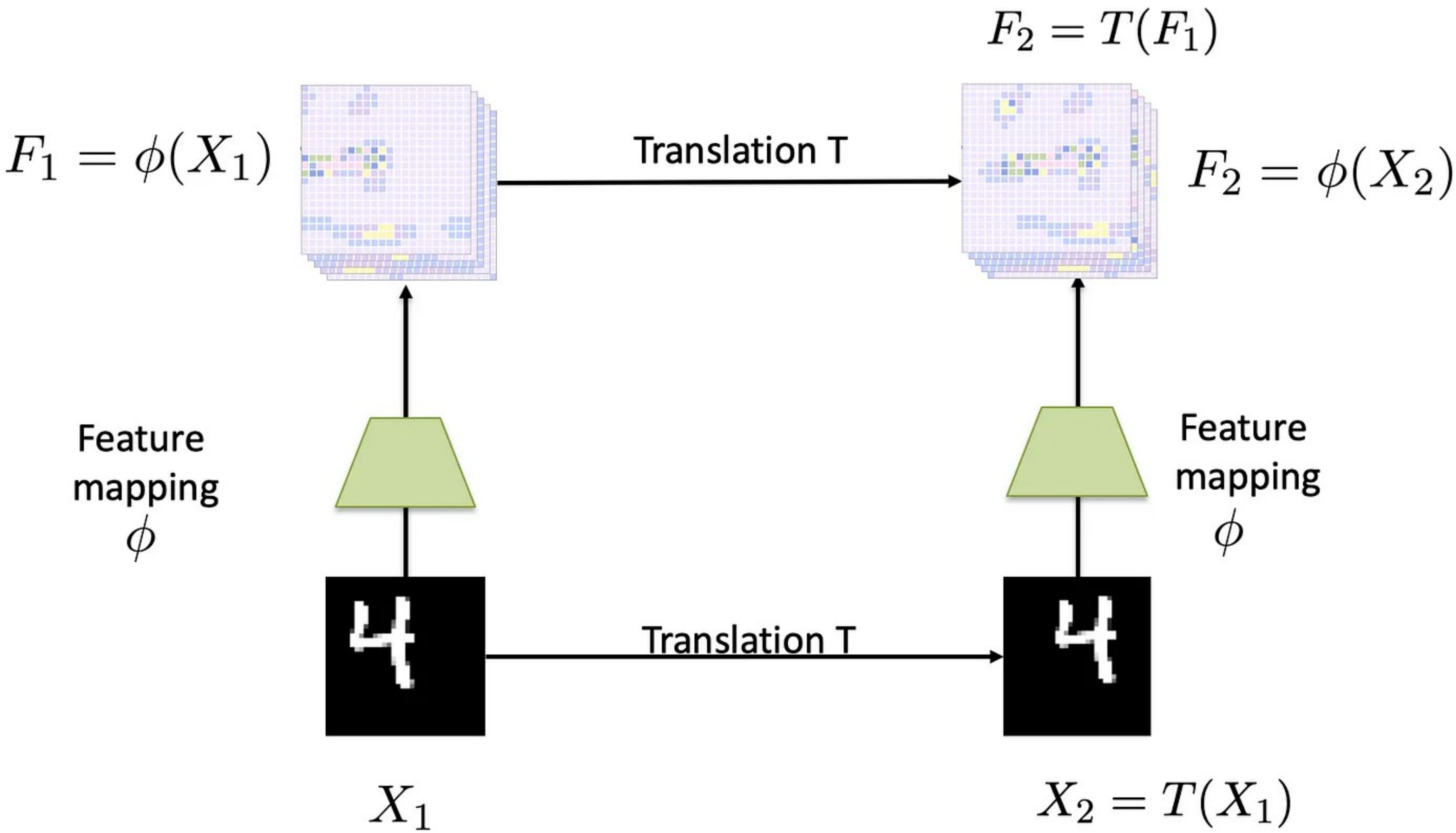
Shift equivariance



Shift equivariance

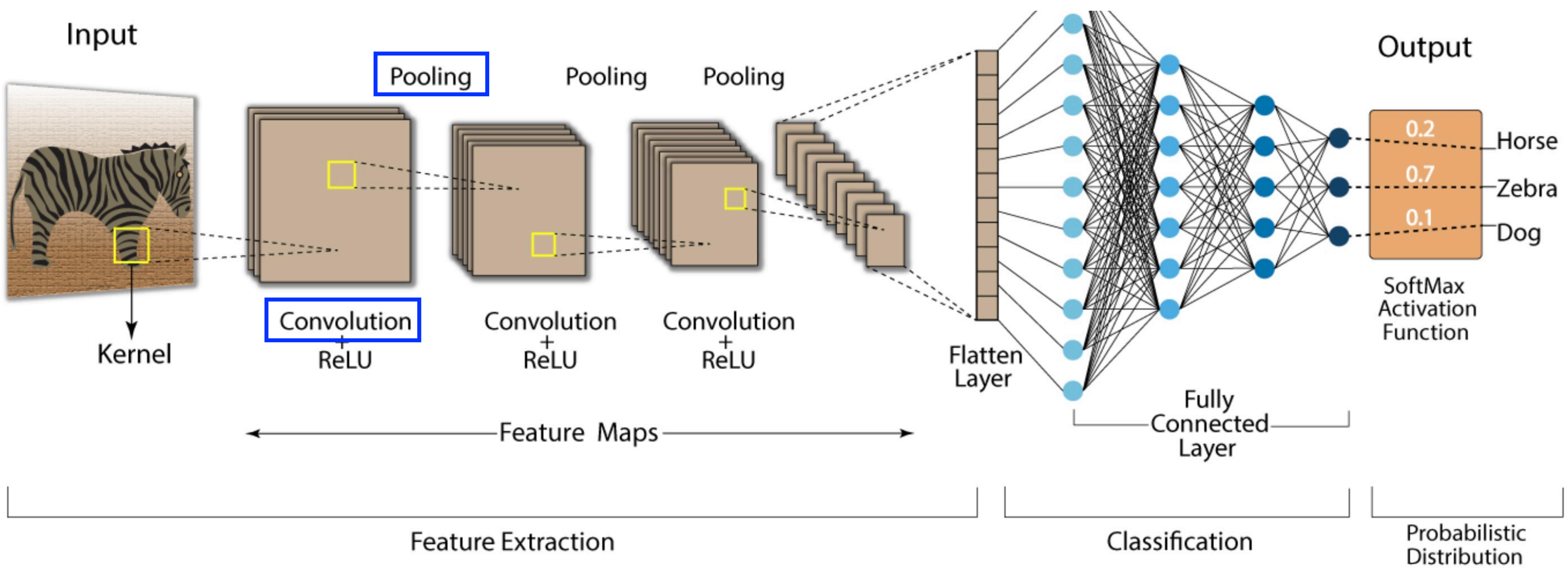


Shift equivariance



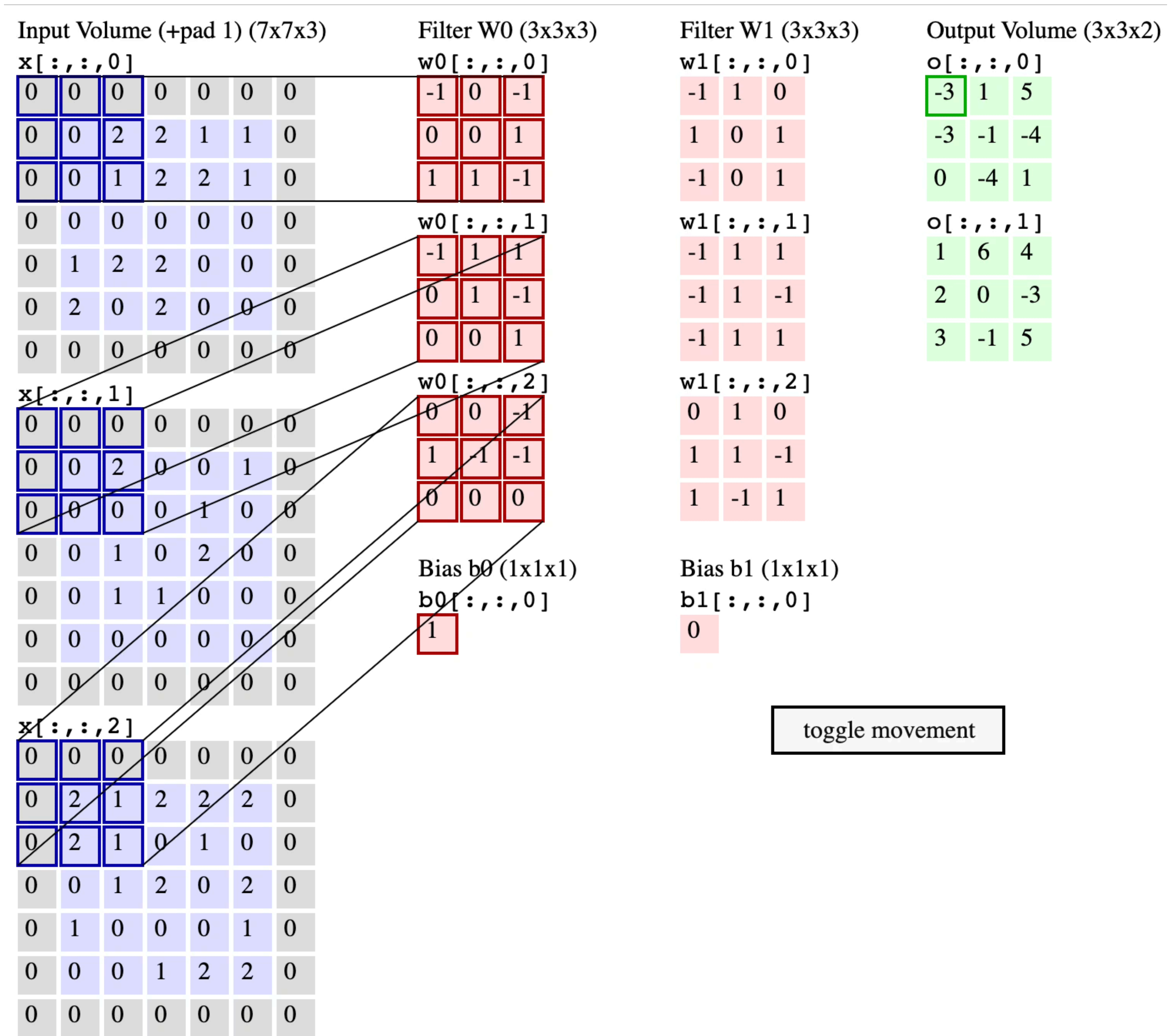
Source : https://chriswolfvision.medium.com/what-is-translation-equivariance-and-why-do-we-use-convolutions-to-get-it-6f18139d4c59

Convolution & Max Pooling invariance ?



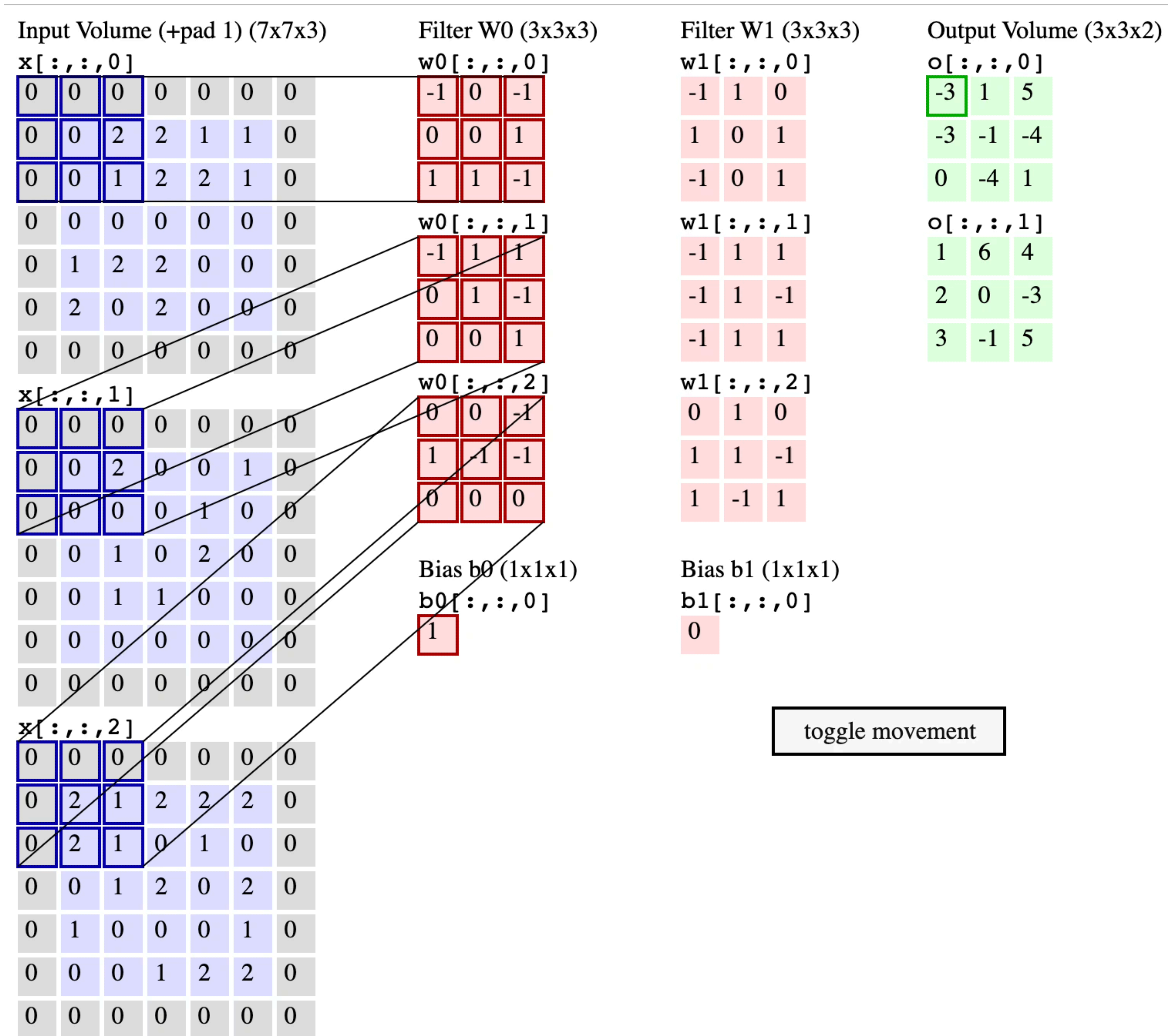
Source : https://developersbreach.com/convolution-neural-network-deep-learning/

Convolutional layers in CNN



Source : https://cs231n.github.io/convolutional-networks/

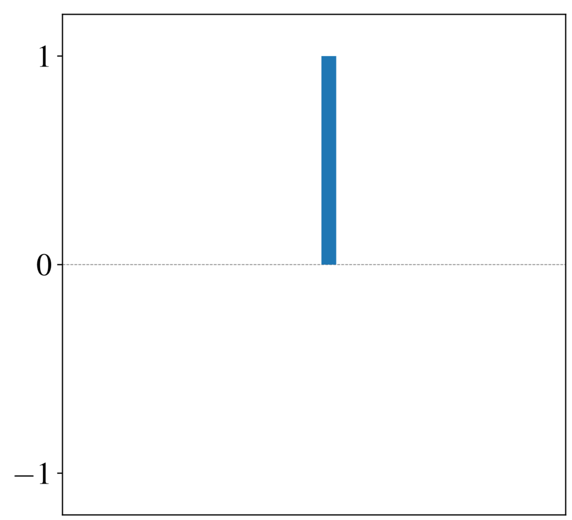
Convolutional layers in CNN



Source : https://cs231n.github.io/convolutional-networks/

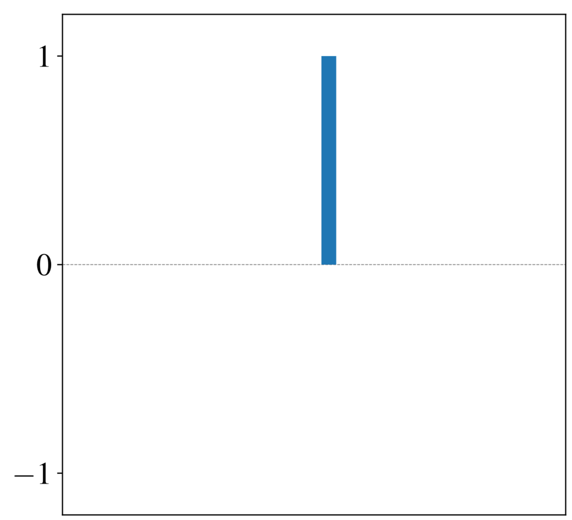
Convolutions are shift-equivariant

Convolutions are shift-equivariant



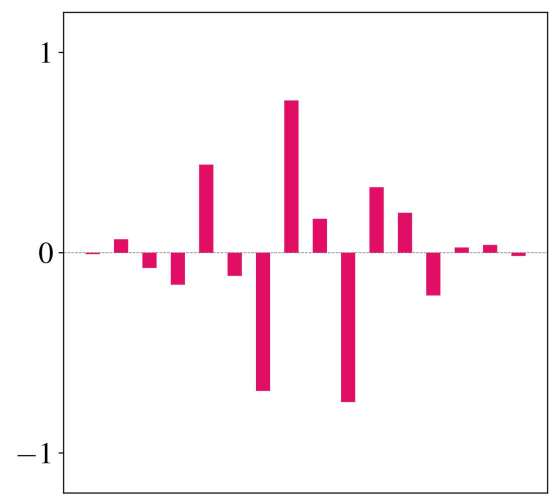
Input signal

Convolutions are shift-equivariant



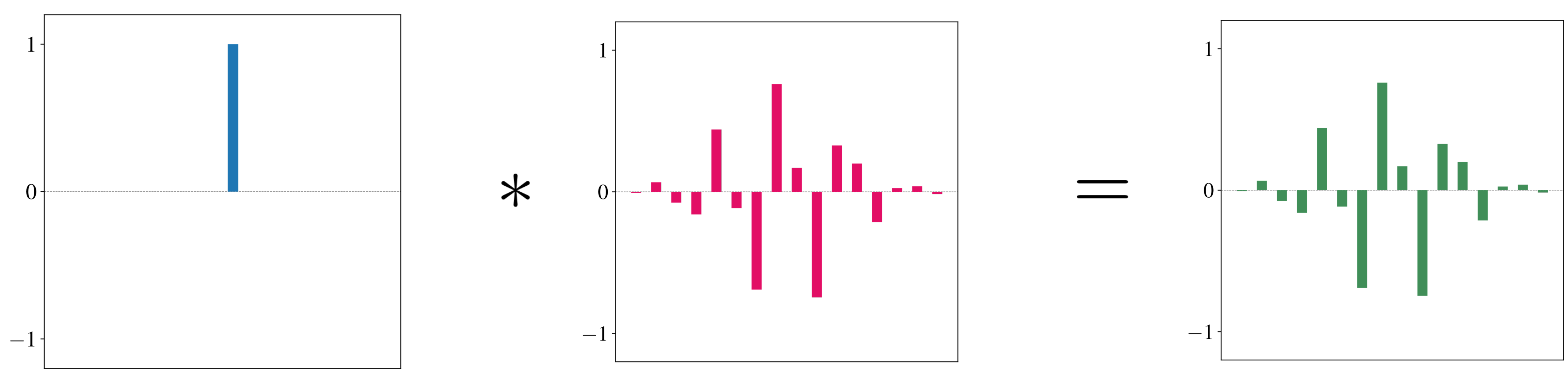
Input signal

$*$



Band-pass
convolution kernel

Convolutions are shift-equivariant

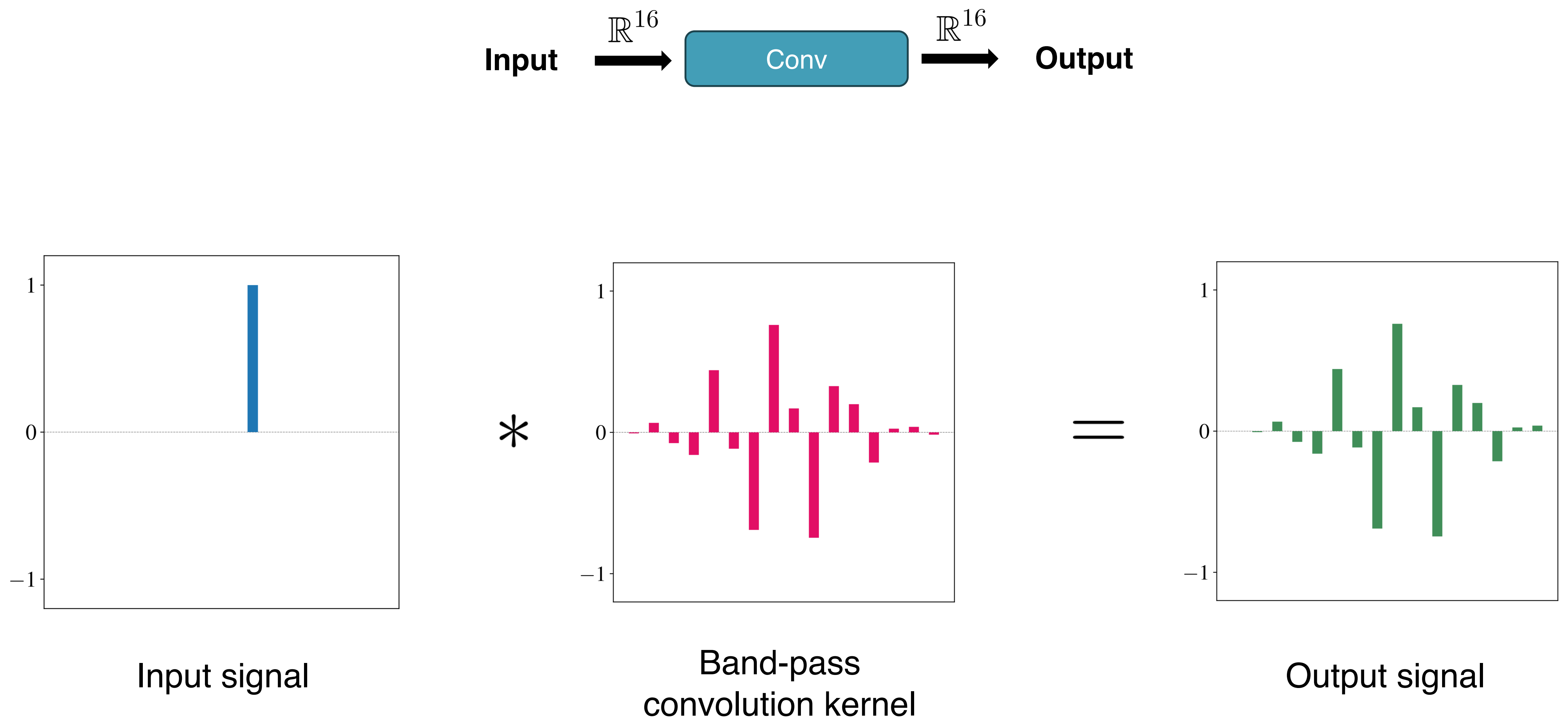


Input signal

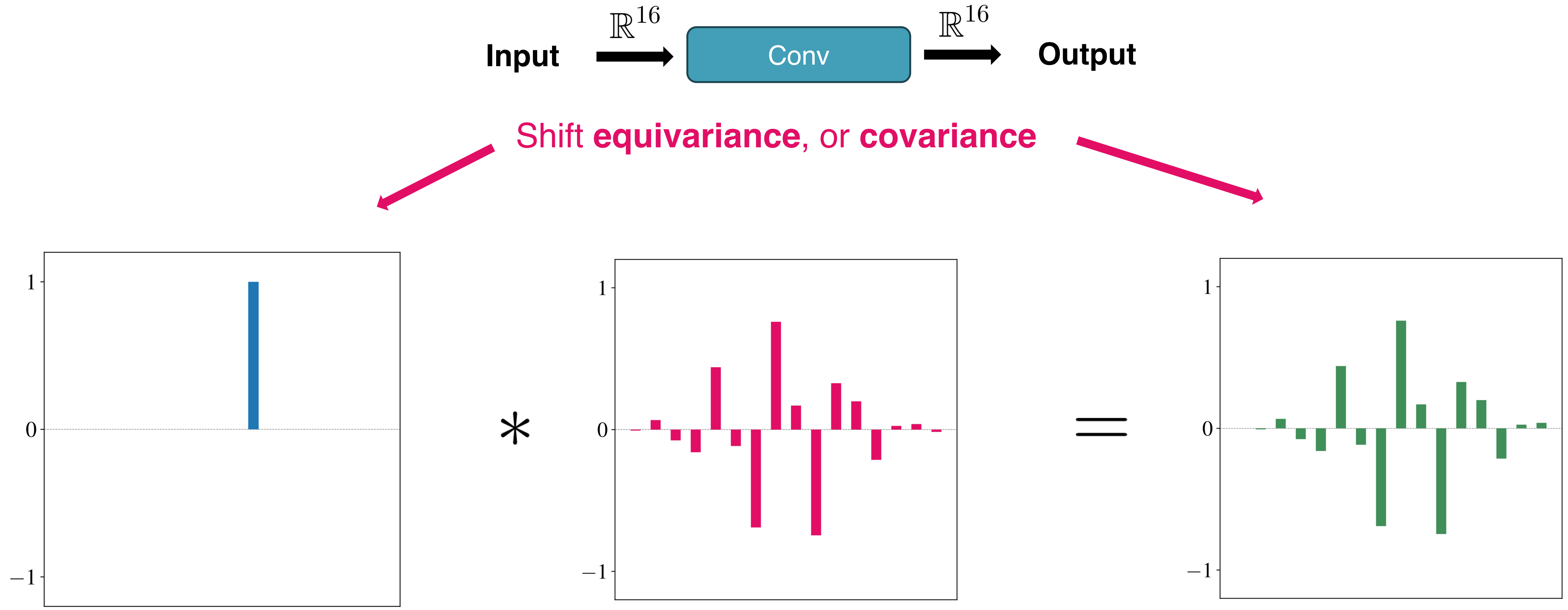
Band-pass
convolution kernel

Output signal

Convolutions are shift-equivariant



Convolutions are shift-equivariant

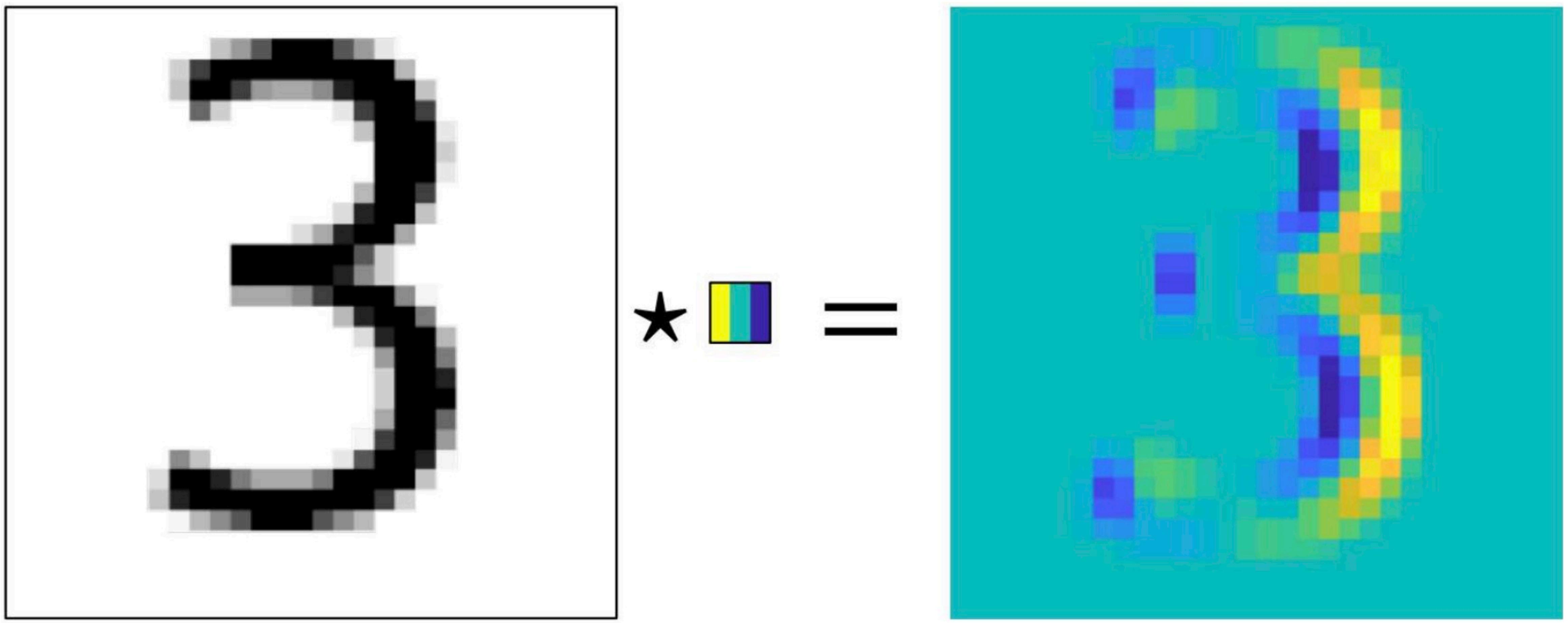


Input signal

Band-pass
convolution kernel

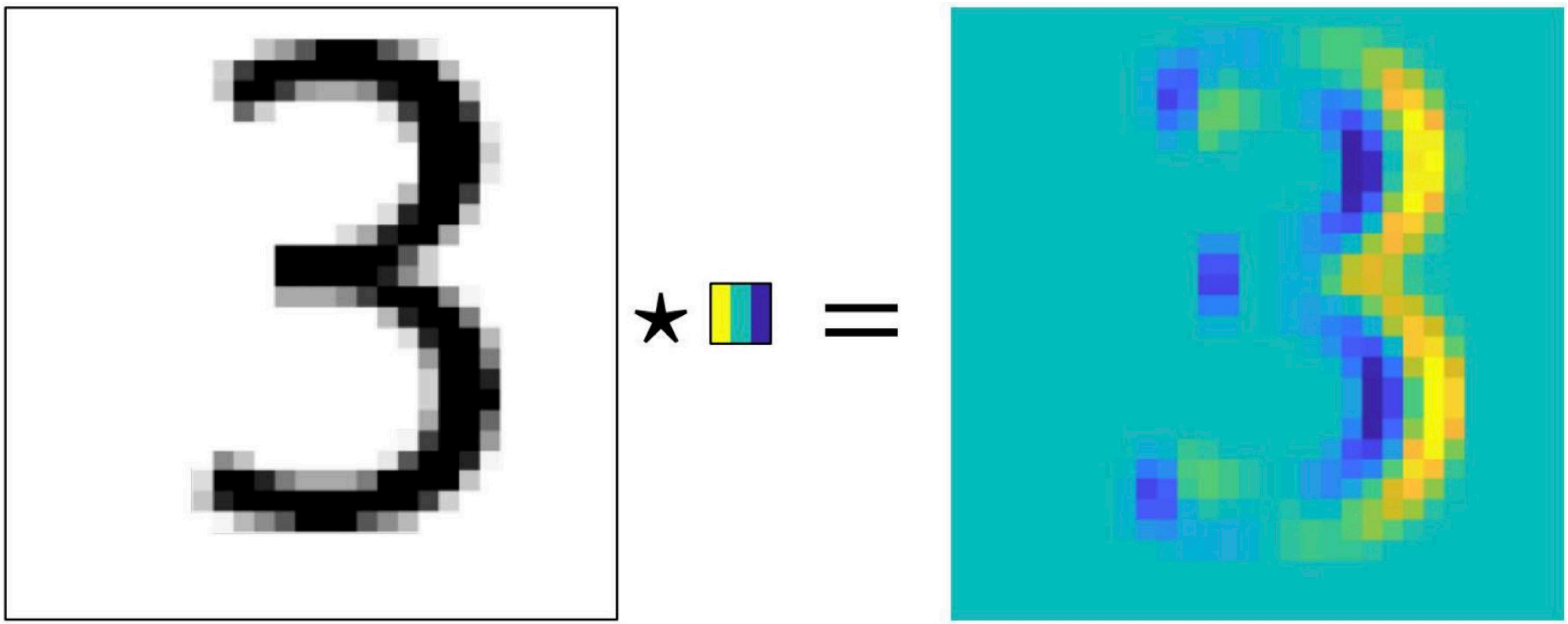
Output signal

Convolutions are shift-equivariant



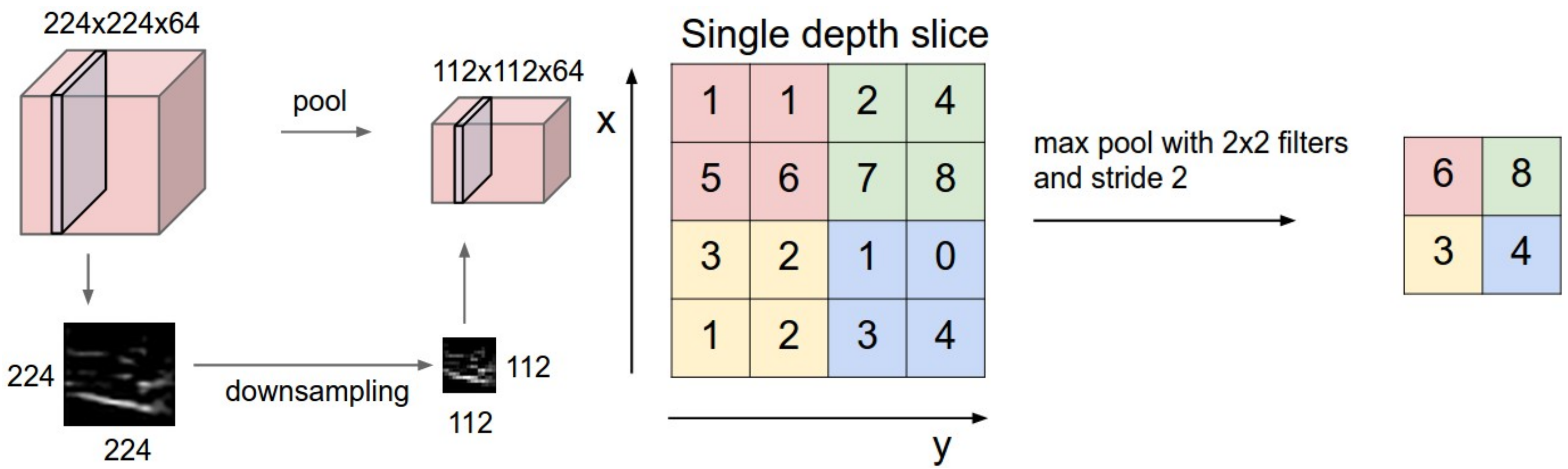
Source : https://www.doc.ic.ac.uk/~bkainz/teaching/DL/notes/equivariance.pdf

Convolutions are shift-equivariant



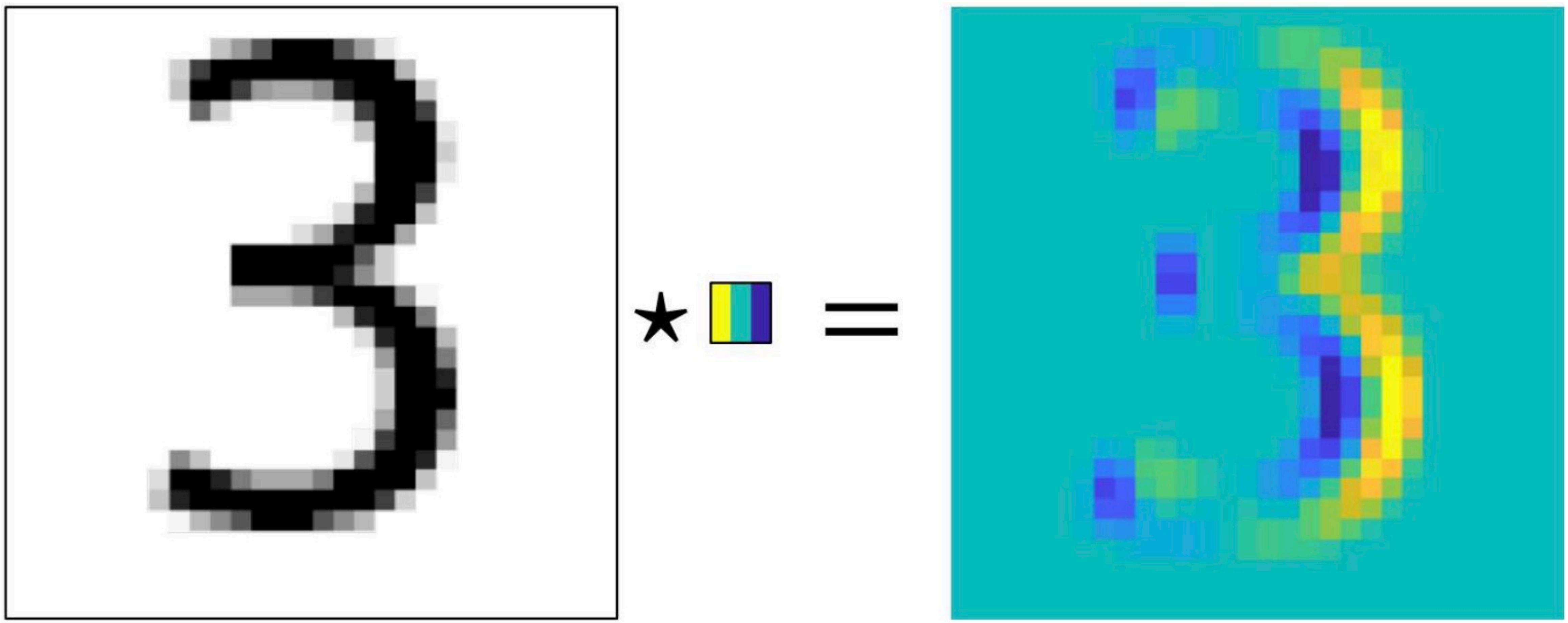
Source : https://www.doc.ic.ac.uk/~bkainz/teaching/DL/notes/equivariance.pdf

Max pooling layers



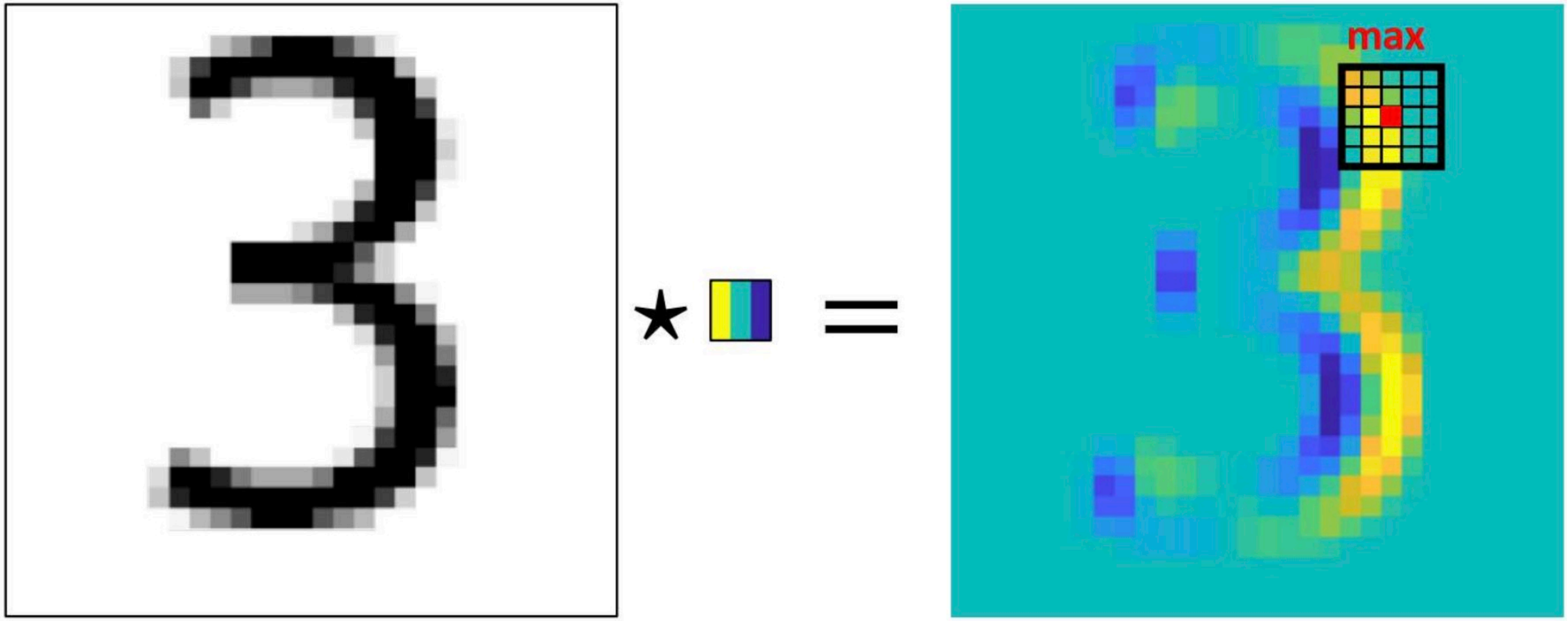
Source : https://cs231n.github.io/convolutional-networks/

Convolutions are followed by a max pooling

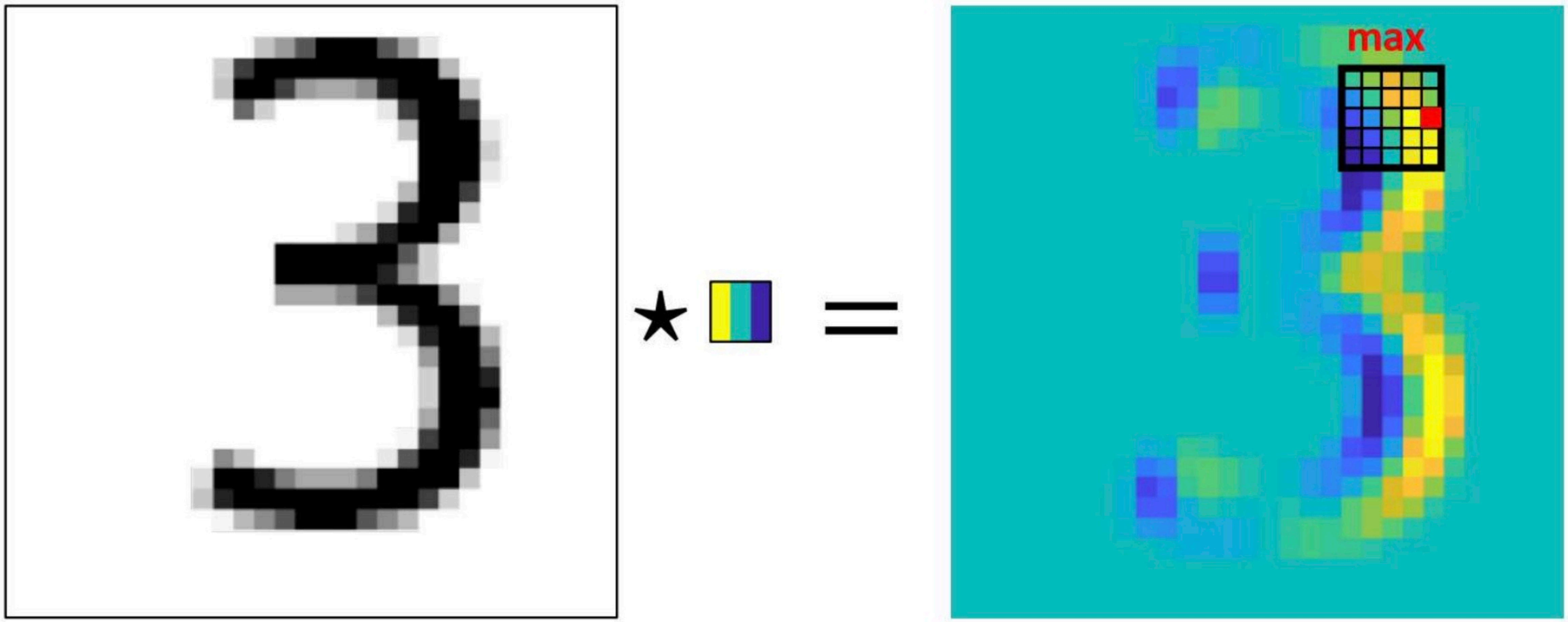


Source : https://www.doc.ic.ac.uk/~bkainz/teaching/DL/notes/equivariance.pdf

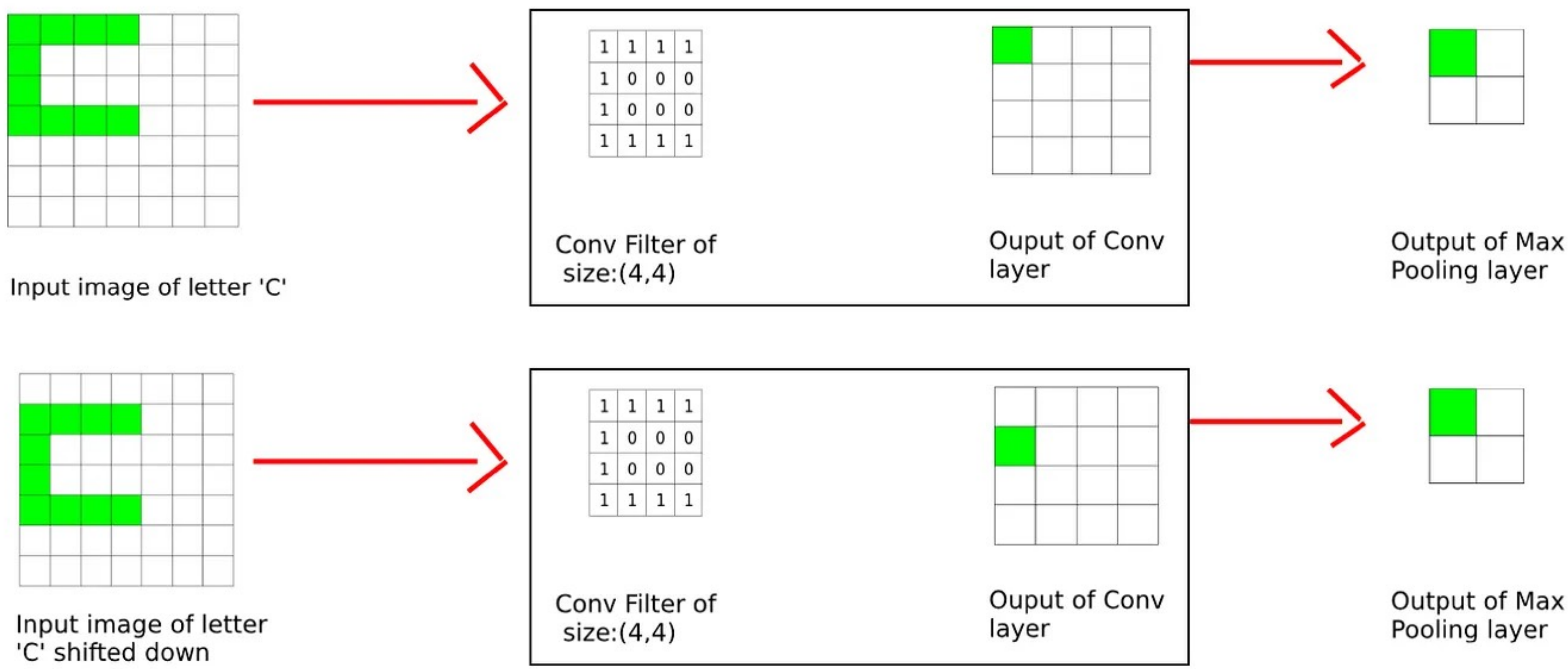
Convolutions are followed by a max pooling



Max pooling builds up shift-invariance



Max pooling builds up shift-invariance



Source : https://divsoni2012.medium.com/translation-invariance-in-convolutional-neural-networks-61d9b6fa03df

Invariance studies in CNN

Invariance studies in CNN

- The **scattering transform** builds shift-invariant feature vectors:

$$\Phi(x) := S_J x = \begin{pmatrix} x \star \phi_{2^J} \\ |x \star \psi_{\lambda_1}| \star \phi_{2^J} \\ ||x \star \psi_{\lambda_1}| \star \psi_{\lambda_2}| \star \phi_{2^J} \\ |||x \star \psi_{\lambda_1}| \star \psi_{\lambda_2}| \star \psi_{\lambda_3}| \star \phi_{2^J} \\ \vdots \end{pmatrix}_{\lambda_1,\lambda_2,\lambda_3,\ldots}$$

Invariance studies in CNN

- The **scattering transform** builds shift-invariant feature vectors:

$$\Phi(x) := S_J x = \begin{pmatrix} x \star \phi_{2^J} \\ |x \star \psi_{\lambda_1}| \star \phi_{2^J} \\ ||x \star \psi_{\lambda_1}| \star \psi_{\lambda_2}| \star \phi_{2^J} \\ |||x \star \psi_{\lambda_1}| \star \psi_{\lambda_2}| \star \psi_{\lambda_3}| \star \phi_{2^J} \\ \vdots \end{pmatrix}_{\lambda_1, \lambda_2, \lambda_3, \ldots}$$

J. Bruna and S. Mallat, “Invariant scattering convolution networks,” IEEE Trans. Pattern Anal. Mach. Intell., vol. 35, no. 8, pp. 1872–1886, 2013.

Invariance studies in CNN

- The **scattering transform** builds shift-invariant feature vectors:

$$\Phi(x) := S_J x = \begin{pmatrix} x \star \phi_{2^J} \\ |x \star \psi_{\lambda_1}| \star \phi_{2^J} \\ ||x \star \psi_{\lambda_1}| \star \psi_{\lambda_2}| \star \phi_{2^J} \\ |||x \star \psi_{\lambda_1}| \star \psi_{\lambda_2}| \star \psi_{\lambda_3}| \star \phi_{2^J} \\ \vdots \end{pmatrix}_{\lambda_1, \lambda_2, \lambda_3, \ldots}$$

- **General deep convolutional neural networks** also become more translation invariant with increasing network depth (proved in the *continuous framework*).

J. Bruna and S. Mallat, "Invariant scattering convolution networks," IEEE Trans. Pattern Anal. Mach. Intell., vol. 35, no. 8, pp. 1872–1886, 2013.

Invariance studies in CNN

- The **scattering transform** builds shift-invariant feature vectors:

$$\Phi(x) := S_J x = \begin{pmatrix} x \star \phi_{2^J} \\ |x \star \psi_{\lambda_1}| \star \phi_{2^J} \\ ||x \star \psi_{\lambda_1}| \star \psi_{\lambda_2}| \star \phi_{2^J} \\ |||x \star \psi_{\lambda_1}| \star \psi_{\lambda_2}| \star \psi_{\lambda_3}| \star \phi_{2^J} \\ \vdots \end{pmatrix}_{\lambda_1, \lambda_2, \lambda_3, \ldots}$$

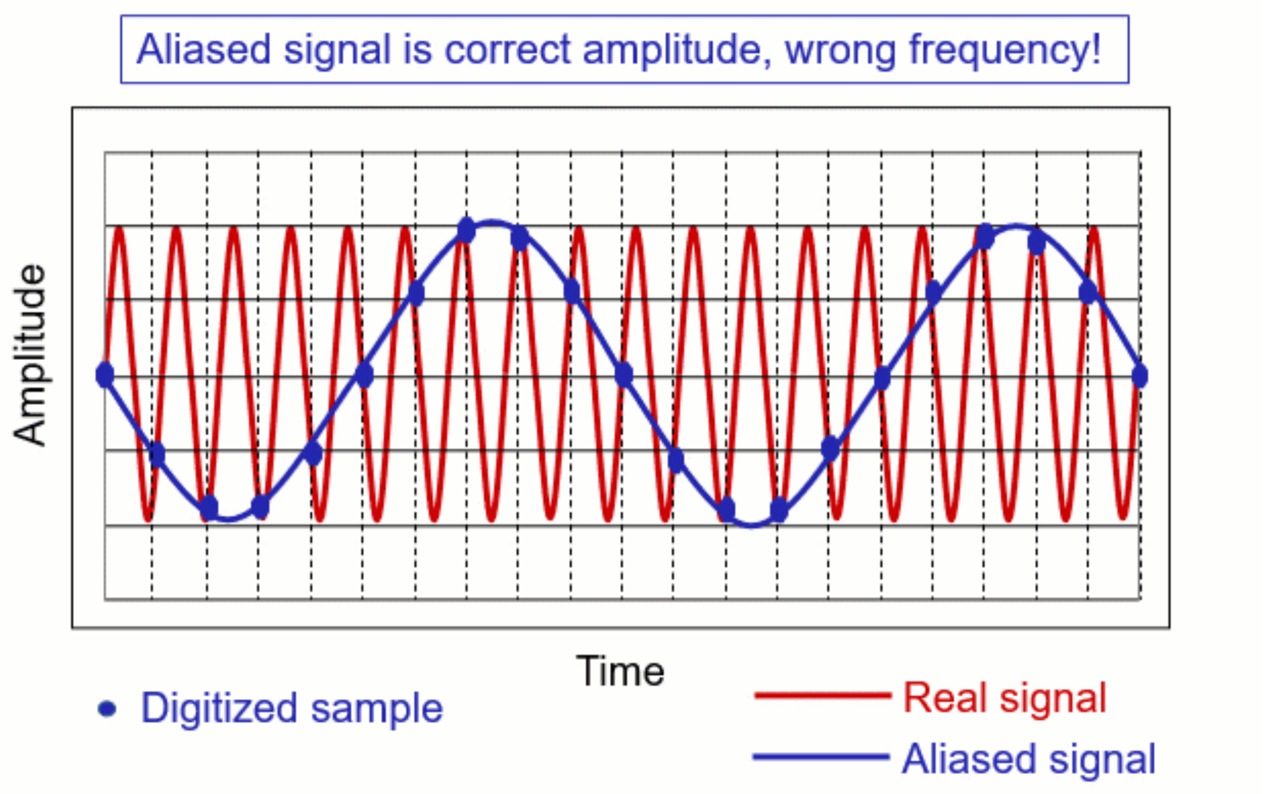
- **General deep convolutional neural networks** also become more translation invariant with increasing network depth (proved in the *continuous framework*).

J. Bruna and S. Mallat, "Invariant scattering convolution networks," IEEE Trans. Pattern Anal. Mach. Intell., vol. 35, no. 8, pp. 1872–1886, 2013.

T. Wiatowski and H. Bölcskei, A Mathematical Theory of Deep Convolutional Neural Networks for Feature Extraction, IEEE Transactions on Information Theory, 64 (2018), pp. 1845–1866

Invariance studies in CNN

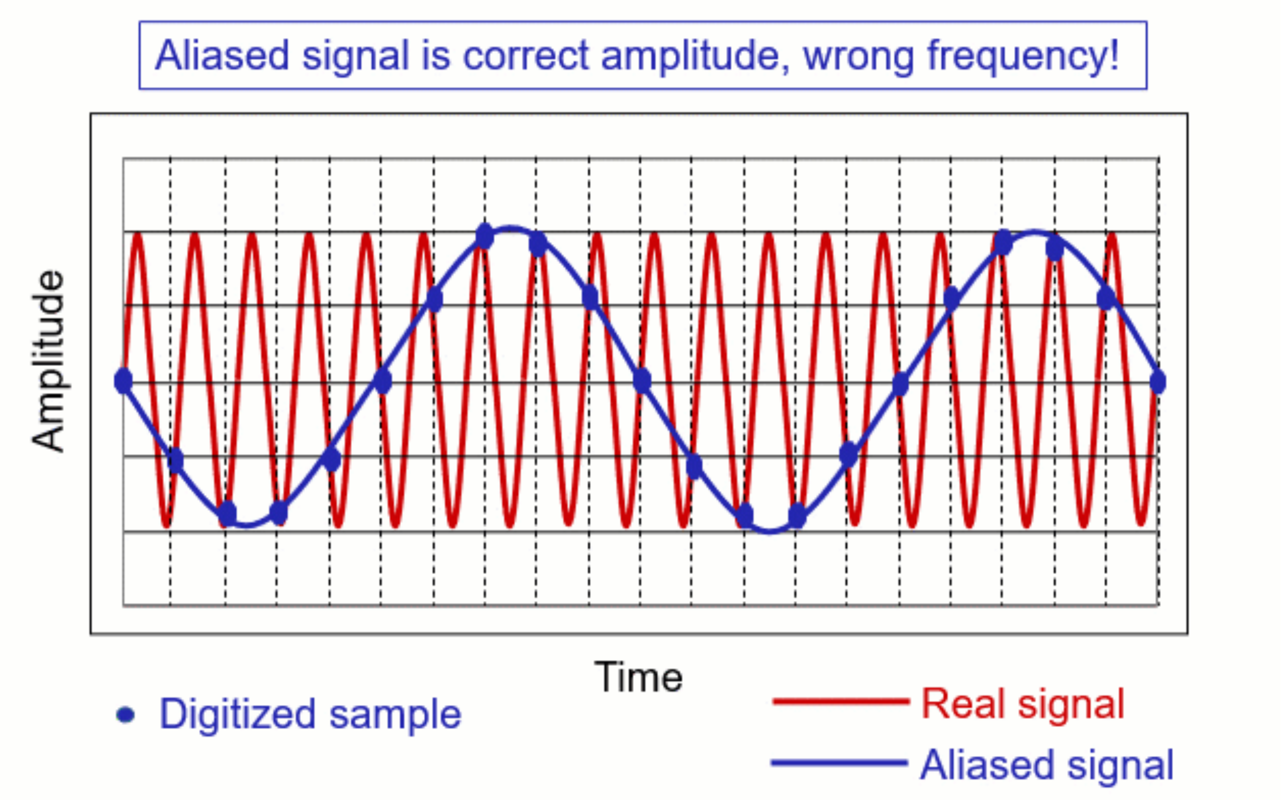
- These results do not fully extend to the ***discrete framework***



Source : https://community.sw.siemens.com/s/article/data-acquisition-anti-aliasing-filters

Invariance studies in CNN

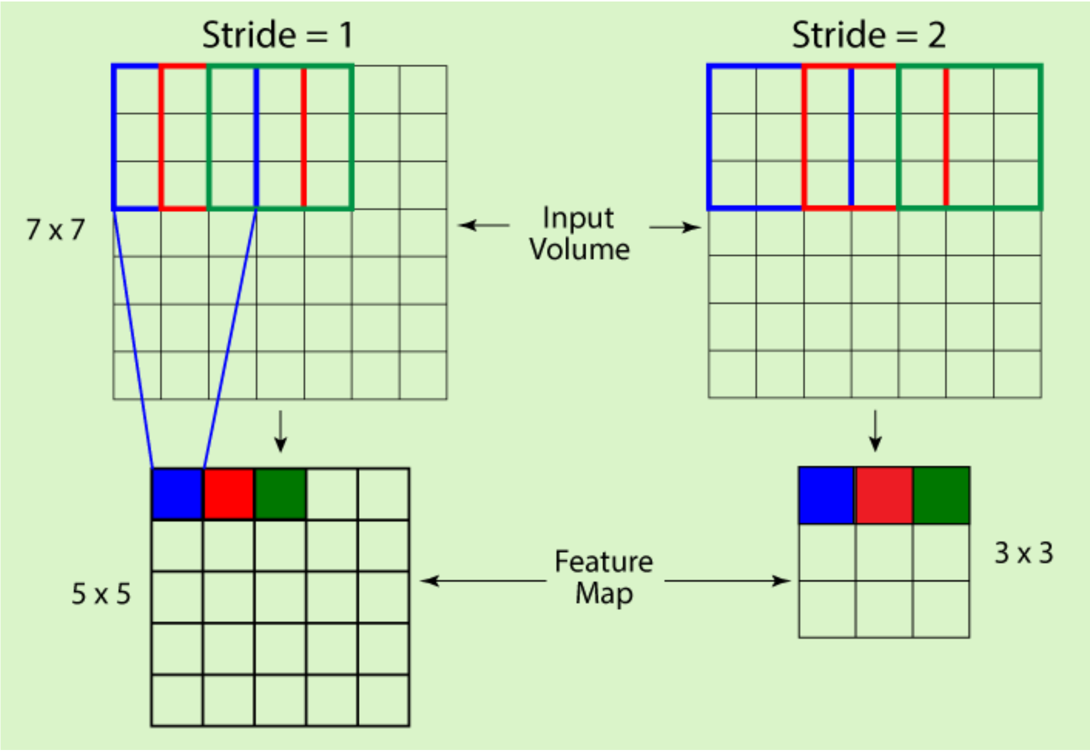
- These results do not fully extend to the ***discrete framework***
- **Strided** convolution and pooling operators may greatly diverge from shift invariance, due to **aliasing** when subsampling high-frequency signals



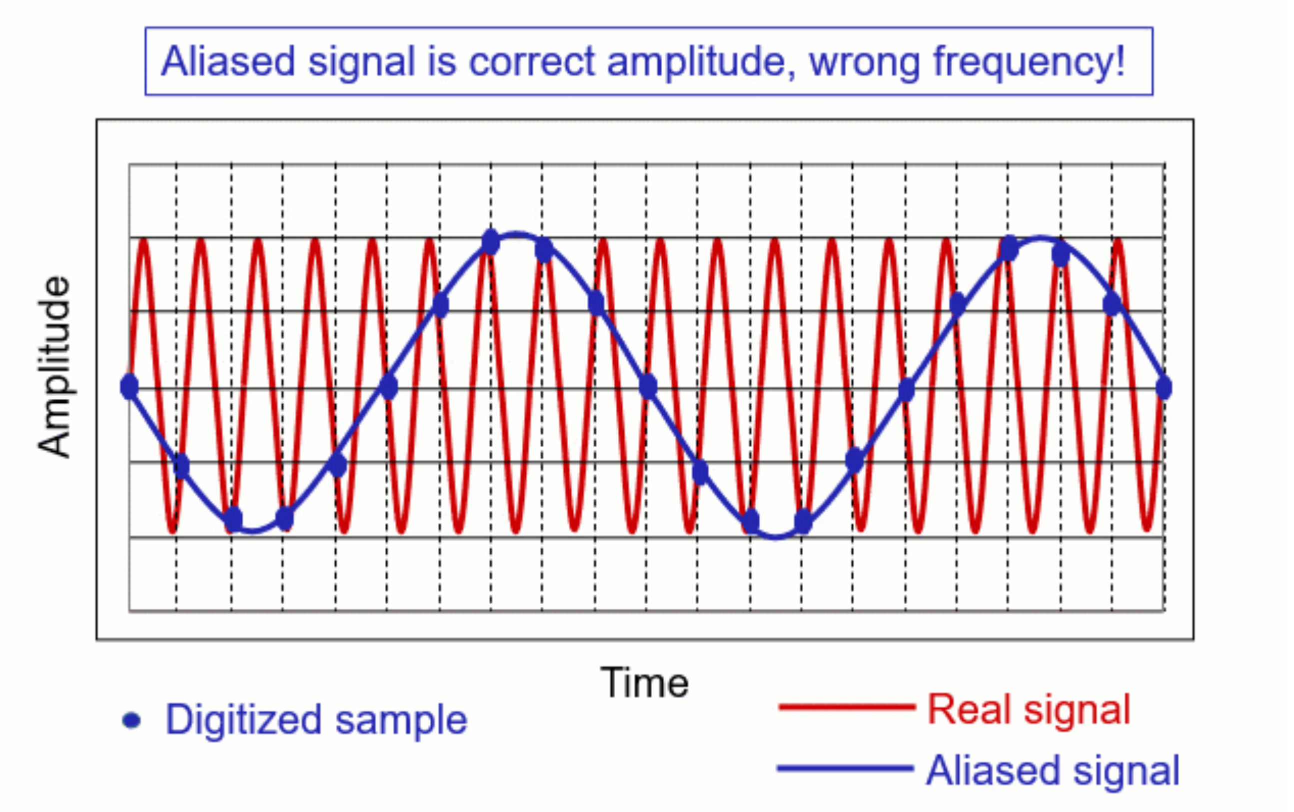
Source : https://community.sw.siemens.com/s/article/data-acquisition-anti-aliasing-filters

Invariance studies in CNN

- These results do not fully extend to the ***discrete framework***
- **Strided** convolution and pooling operators may greatly diverge from shift invariance, due to **aliasing** when subsampling high-frequency signals



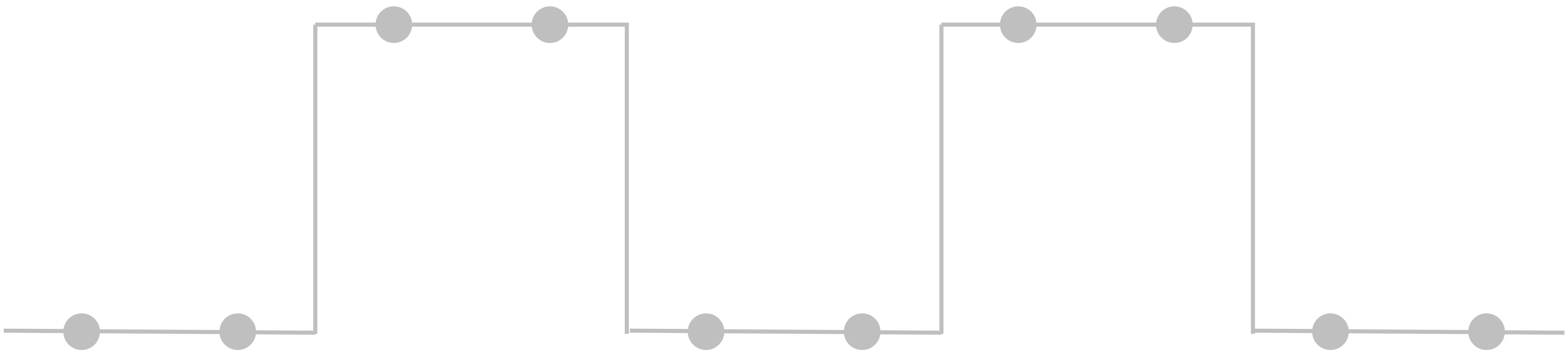
Source : https://developersbreach.com/convolution-neural-network-deep-learning/



Source : https://community.sw.siemens.com/s/article/data-acquisition-anti-aliasing-filters

Invariance studies in CNN

- These results do not fully extend to the ***discrete framework***
- **Strided** convolution and pooling operators may greatly diverge from shift invariance, due to **aliasing** when subsampling high-frequency signals

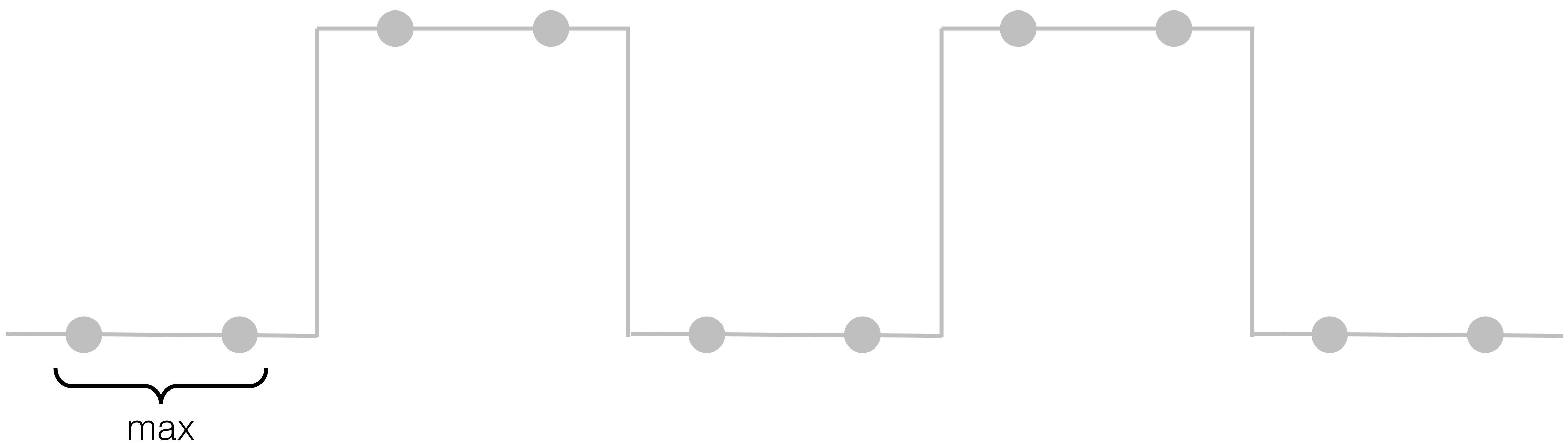


R. Zhang, Making Convolutional Networks Shift-Invariant Again, in International Conference on Machine Learning, 2019.

A. Azulay and Y. Weiss, Why do deep convolutional networks generalize so poorly to small image transformations?, Journal of Machine Learning Research, 2019.

Invariance studies in CNN

- These results do not fully extend to the ***discrete framework***
- **Strided** convolution and pooling operators may greatly diverge from shift invariance, due to **aliasing** when subsampling high-frequency signals



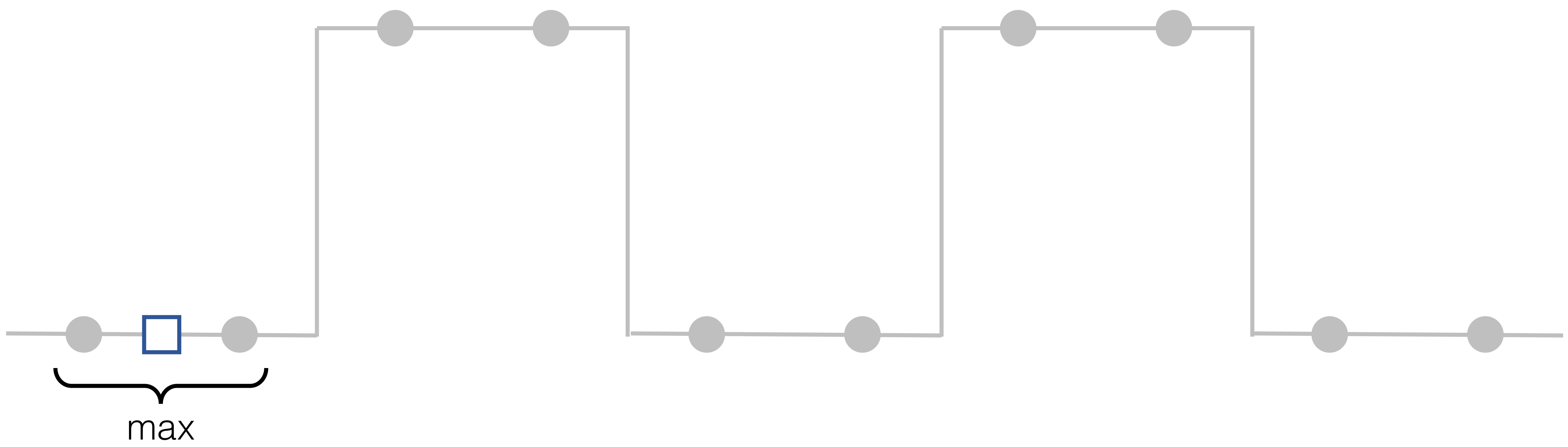
R. Zhang, Making Convolutional Networks Shift-Invariant Again, in International Conference on Machine Learning, 2019.

A. Azulay and Y. Weiss, Why do deep convolutional networks generalize so poorly to small image transformations?, Journal of Machine Learning Research, 2019.

Invariance studies in CNN

- These results do not fully extend to the ***discrete framework***

- **Strided** convolution and pooling operators may greatly diverge from shift invariance, due to **aliasing** when subsampling high-frequency signals

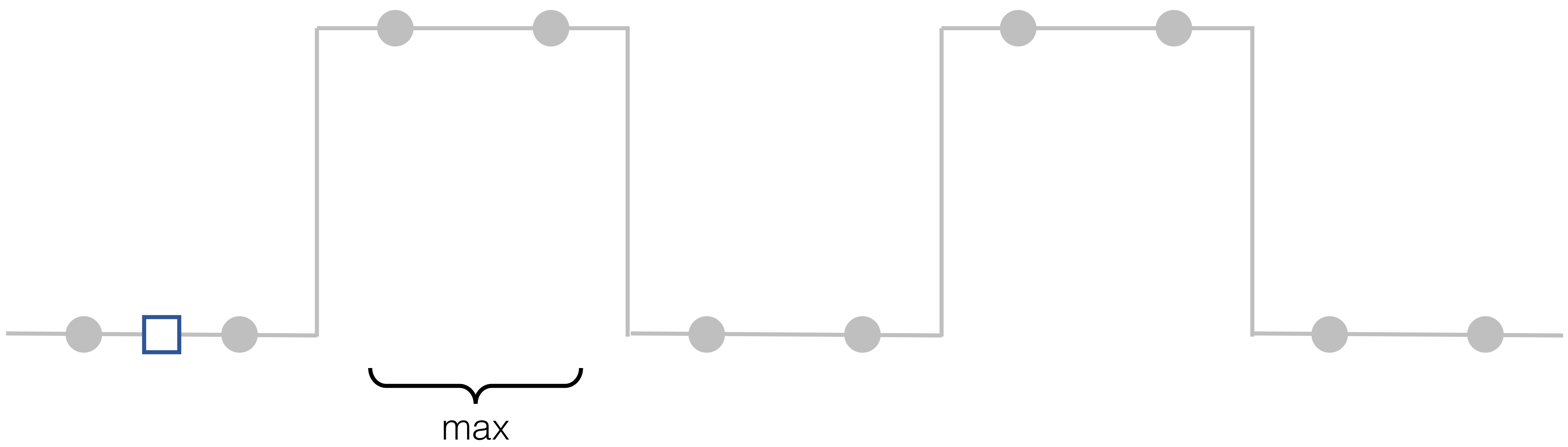


R. Zhang, Making Convolutional Networks Shift-Invariant Again, in International Conference on Machine Learning, 2019.

A. Azulay and Y. Weiss, Why do deep convolutional networks generalize so poorly to small image transformations?, Journal of Machine Learning Research, 2019.

Invariance studies in CNN

- These results do not fully extend to the ***discrete framework***
- **Strided** convolution and pooling operators may greatly diverge from shift invariance, due to **aliasing** when subsampling high-frequency signals

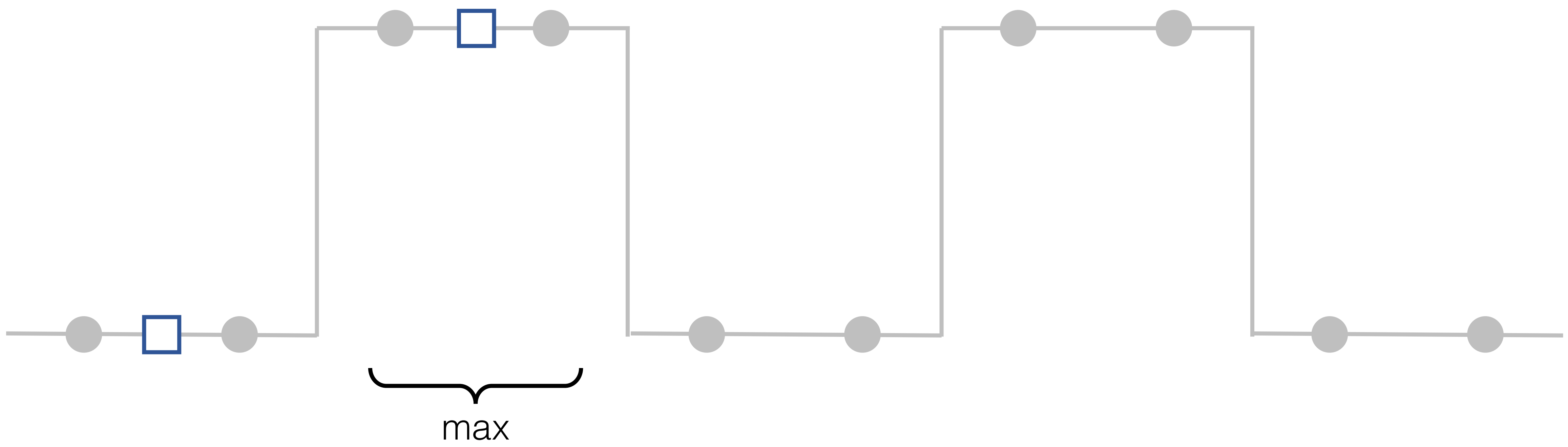


R. Zhang, Making Convolutional Networks Shift-Invariant Again, in International Conference on Machine Learning, 2019.

A. Azulay and Y. Weiss, Why do deep convolutional networks generalize so poorly to small image transformations?, Journal of Machine Learning Research, 2019.

Invariance studies in CNN

- These results do not fully extend to the ***discrete framework***
- **Strided** convolution and pooling operators may greatly diverge from shift invariance, due to **aliasing** when subsampling high-frequency signals

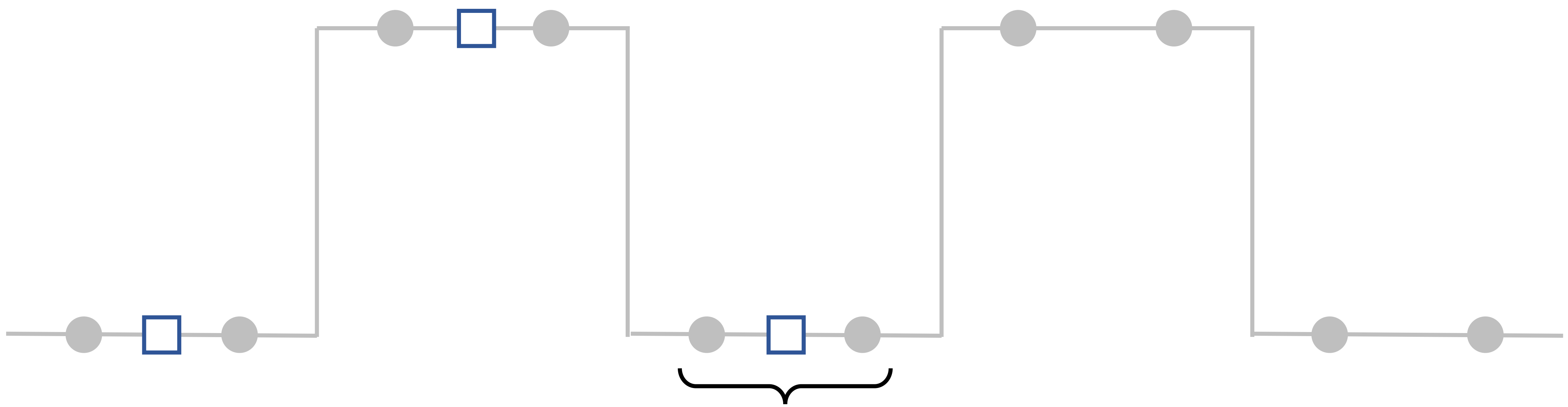


R. Zhang, Making Convolutional Networks Shift-Invariant Again, in International Conference on Machine Learning, 2019.

A. Azulay and Y. Weiss, Why do deep convolutional networks generalize so poorly to small image transformations?, Journal of Machine Learning Research, 2019.

Invariance studies in CNN

- These results do not fully extend to the ***discrete framework***
- **Strided** convolution and pooling operators may greatly diverge from shift invariance, due to **aliasing** when subsampling high-frequency signals



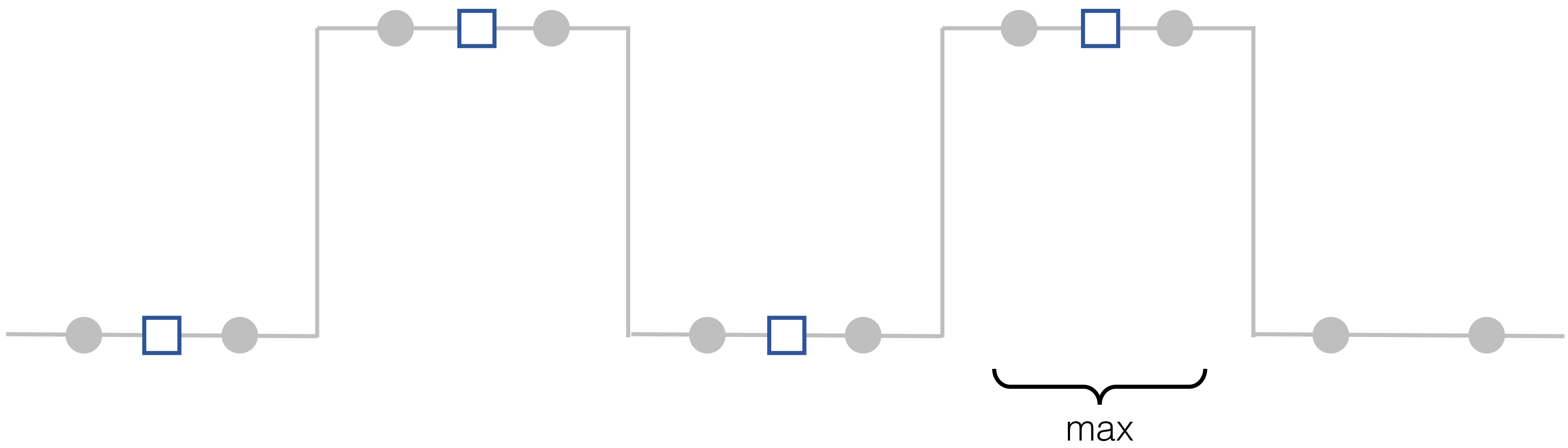
R. Zhang, Making Convolutional Networks Shift-Invariant Again, in International Conference on Machine Learning, 2019.

A. Azulay and Y. Weiss, Why do deep convolutional networks generalize so poorly to small image transformations?, Journal of Machine Learning Research, 2019.

Invariance studies in CNN

- These results do not fully extend to the ***discrete framework***

- **Strided** convolution and pooling operators may greatly diverge from shift invariance, due to **aliasing** when subsampling high-frequency signals

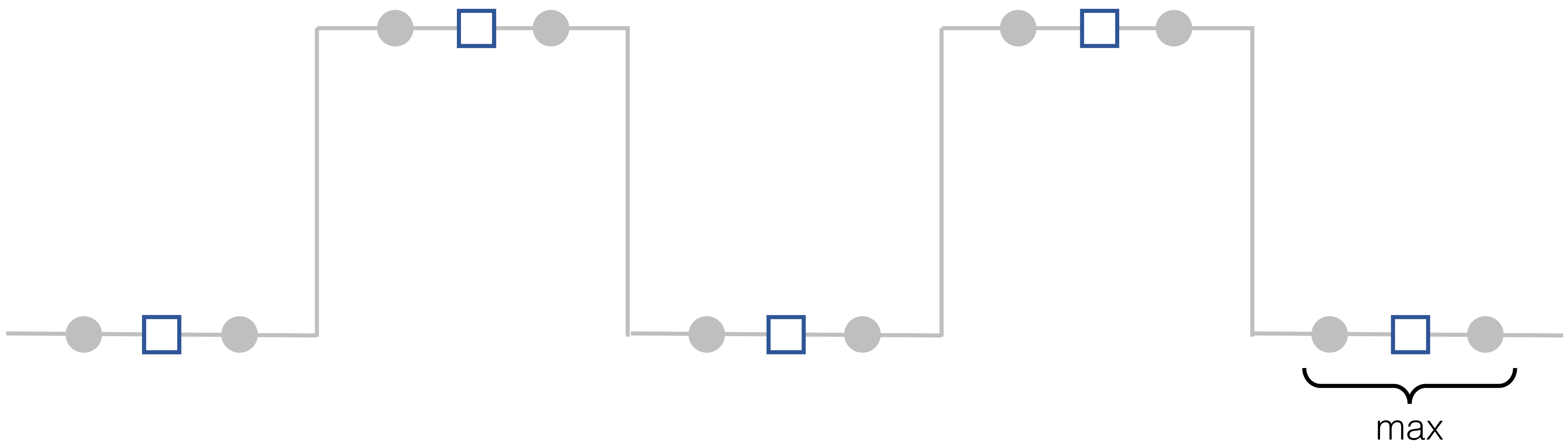


R. Zhang, Making Convolutional Networks Shift-Invariant Again, in International Conference on Machine Learning, 2019.

A. Azulay and Y. Weiss, Why do deep convolutional networks generalize so poorly to small image transformations?, Journal of Machine Learning Research, 2019.

Invariance studies in CNN

- These results do not fully extend to the ***discrete framework***
- **Strided** convolution and pooling operators may greatly diverge from shift invariance, due to **aliasing** when subsampling high-frequency signals

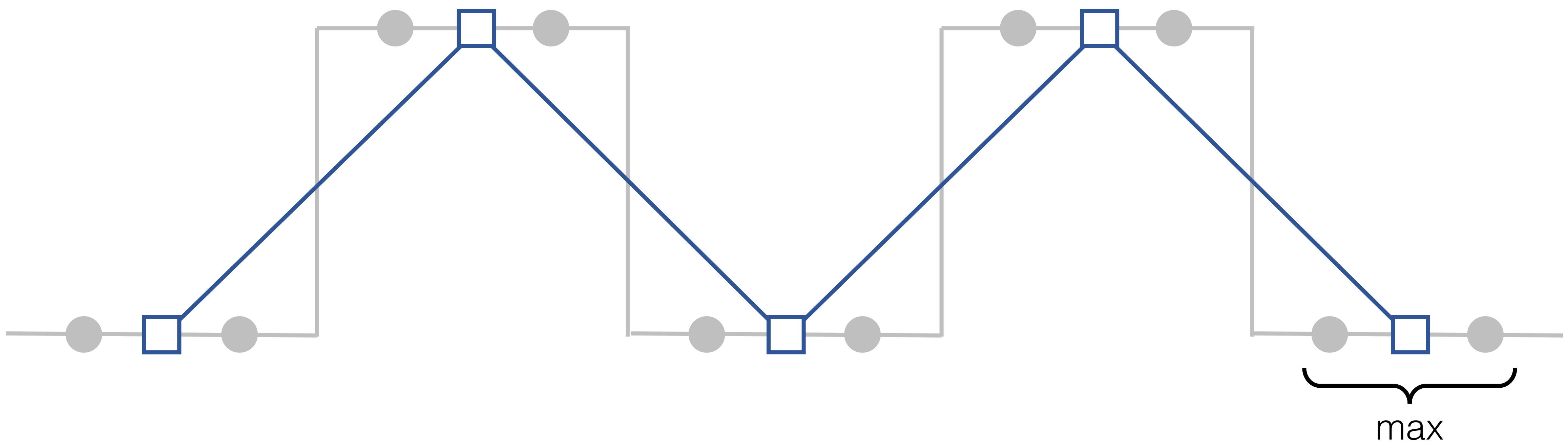


R. Zhang, Making Convolutional Networks Shift-Invariant Again, in International Conference on Machine Learning, 2019.

A. Azulay and Y. Weiss, Why do deep convolutional networks generalize so poorly to small image transformations?, Journal of Machine Learning Research, 2019.

Invariance studies in CNN

- These results do not fully extend to the ***discrete framework***
- **Strided** convolution and pooling operators may greatly diverge from shift invariance, due to **aliasing** when subsampling high-frequency signals

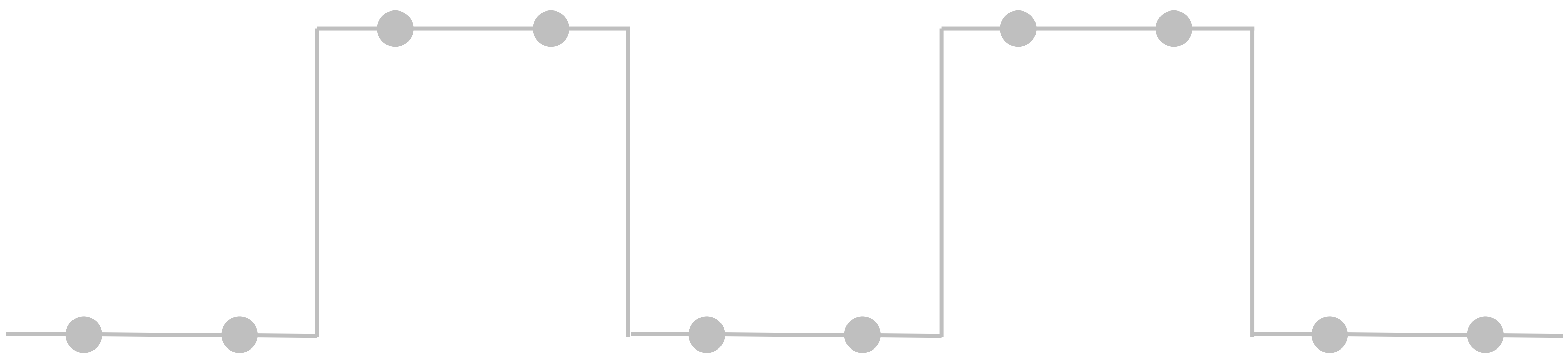


R. Zhang, Making Convolutional Networks Shift-Invariant Again, in International Conference on Machine Learning, 2019.

A. Azulay and Y. Weiss, Why do deep convolutional networks generalize so poorly to small image transformations?, Journal of Machine Learning Research, 2019.

Invariance studies in CNN

- These results do not fully extend to the ***discrete framework***
- **Strided** convolution and pooling operators may greatly diverge from shift invariance, due to **aliasing** when subsampling high-frequency signals

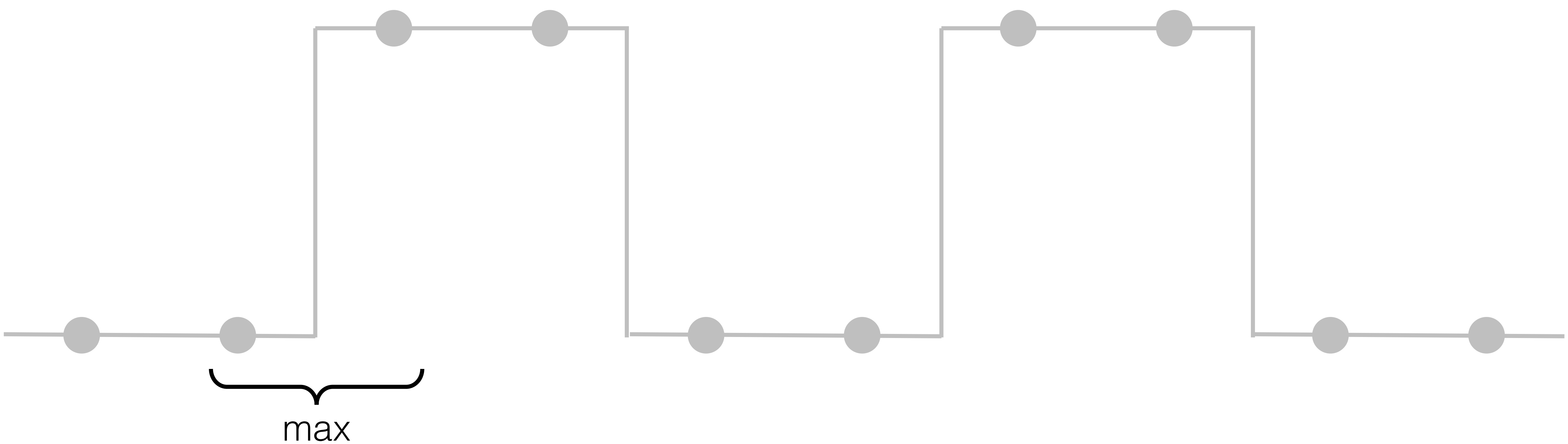


R. Zhang, Making Convolutional Networks Shift-Invariant Again, in International Conference on Machine Learning, 2019.

A. Azulay and Y. Weiss, Why do deep convolutional networks generalize so poorly to small image transformations?, Journal of Machine Learning Research, 2019.

Invariance studies in CNN

- These results do not fully extend to the ***discrete framework***
- **Strided** convolution and pooling operators may greatly diverge from shift invariance, due to **aliasing** when subsampling high-frequency signals

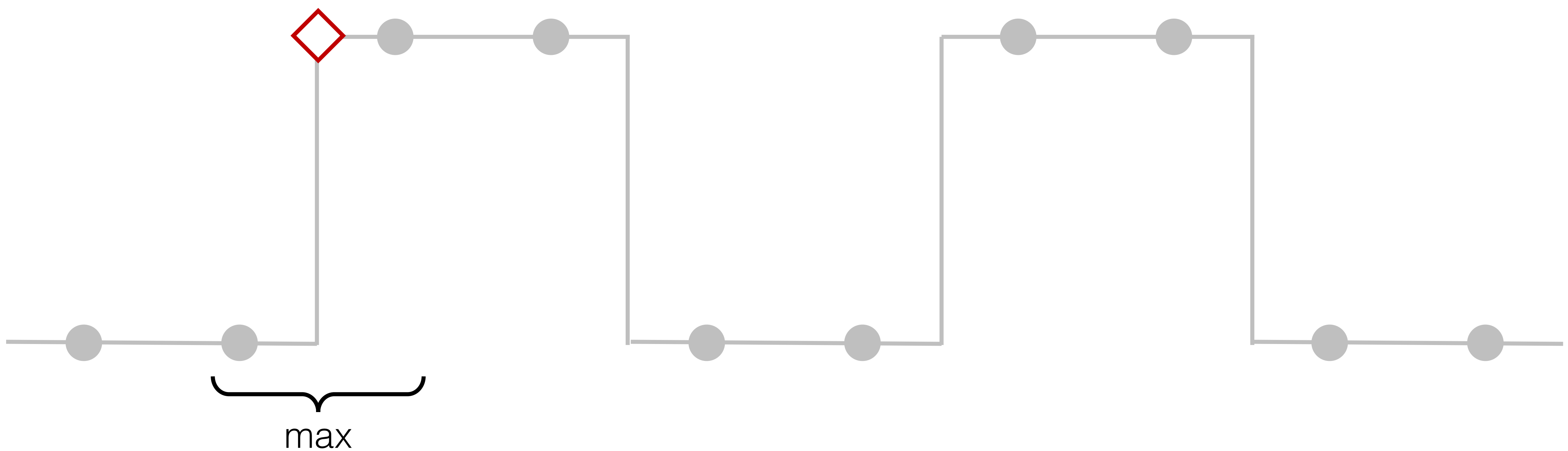


R. Zhang, Making Convolutional Networks Shift-Invariant Again, in International Conference on Machine Learning, 2019.

A. Azulay and Y. Weiss, Why do deep convolutional networks generalize so poorly to small image transformations?, Journal of Machine Learning Research, 2019.

Invariance studies in CNN

- These results do not fully extend to the ***discrete framework***
- **Strided** convolution and pooling operators may greatly diverge from shift invariance, due to **aliasing** when subsampling high-frequency signals

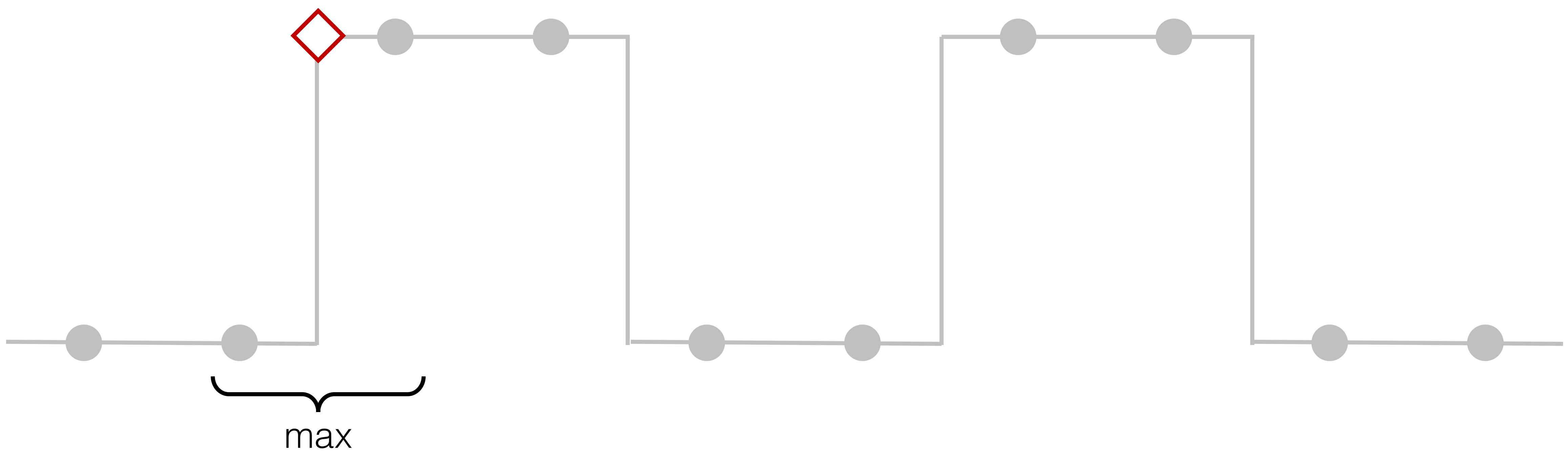


R. Zhang, Making Convolutional Networks Shift-Invariant Again, in International Conference on Machine Learning, 2019.

A. Azulay and Y. Weiss, Why do deep convolutional networks generalize so poorly to small image transformations?, Journal of Machine Learning Research, 2019.

Invariance studies in CNN

- These results do not fully extend to the ***discrete framework***
- **Strided** convolution and pooling operators may greatly diverge from shift invariance, due to **aliasing** when subsampling high-frequency signals

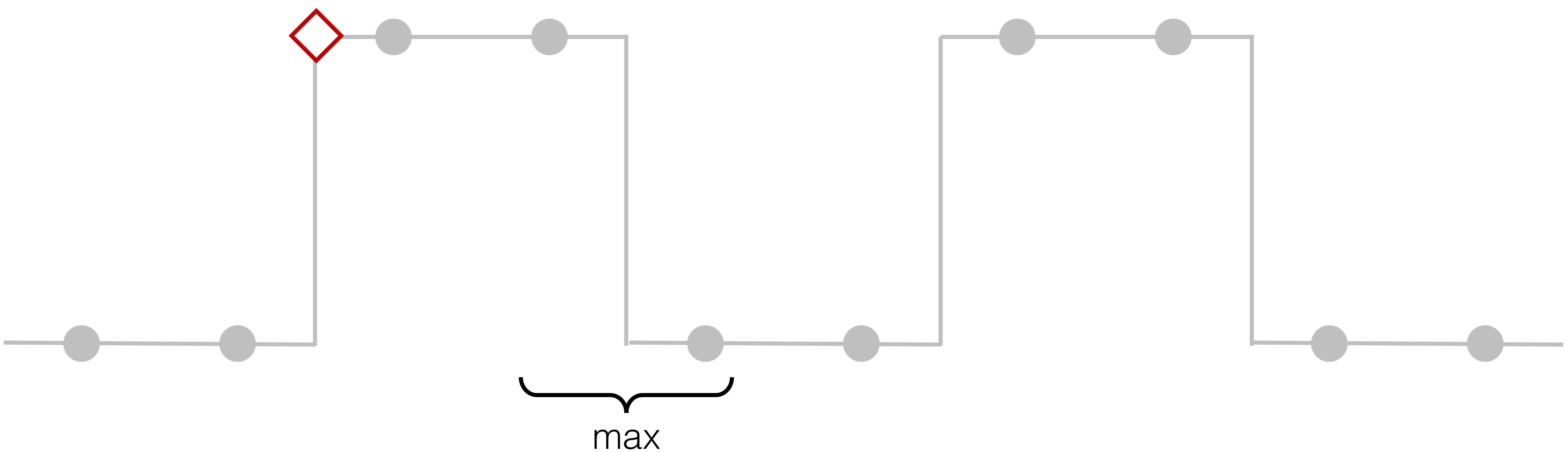


R. Zhang, Making Convolutional Networks Shift-Invariant Again, in International Conference on Machine Learning, 2019.

A. Azulay and Y. Weiss, Why do deep convolutional networks generalize so poorly to small image transformations?, Journal of Machine Learning Research, 2019.

Invariance studies in CNN

- These results do not fully extend to the ***discrete framework***
- **Strided** convolution and pooling operators may greatly diverge from shift invariance, due to **aliasing** when subsampling high-frequency signals

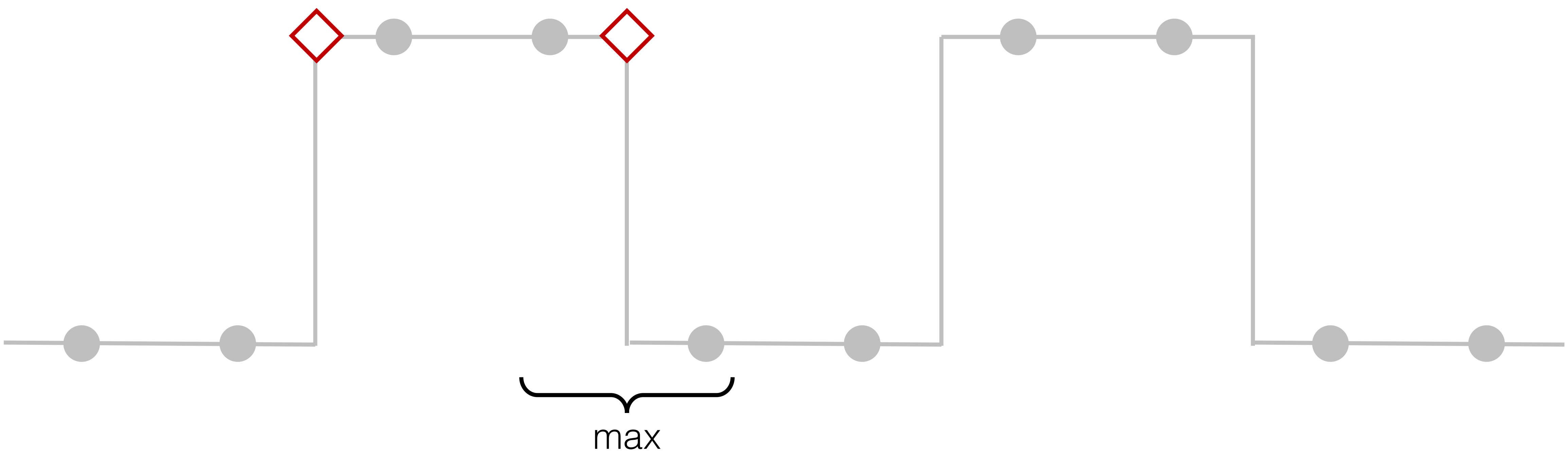


R. Zhang, Making Convolutional Networks Shift-Invariant Again, in International Conference on Machine Learning, 2019.

A. Azulay and Y. Weiss, Why do deep convolutional networks generalize so poorly to small image transformations?, Journal of Machine Learning Research, 2019.

Invariance studies in CNN

- These results do not fully extend to the ***discrete framework***
- **Strided** convolution and pooling operators may greatly diverge from shift invariance, due to **aliasing** when subsampling high-frequency signals

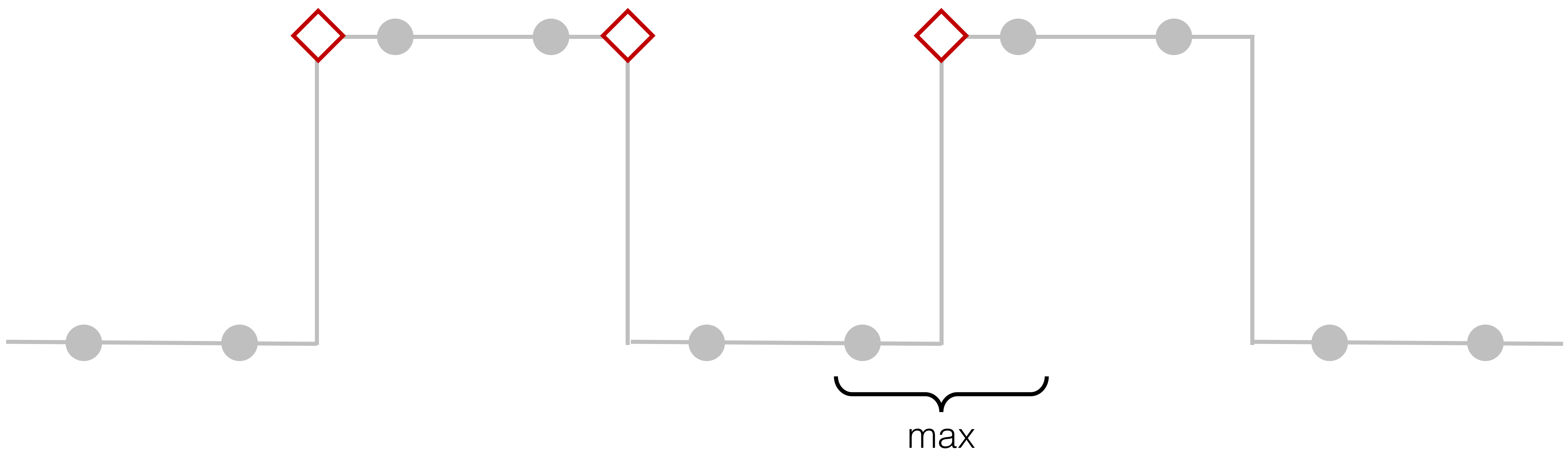


R. Zhang, Making Convolutional Networks Shift-Invariant Again, in International Conference on Machine Learning, 2019.

A. Azulay and Y. Weiss, Why do deep convolutional networks generalize so poorly to small image transformations?, Journal of Machine Learning Research, 2019.

Invariance studies in CNN

- These results do not fully extend to the ***discrete framework***
- **Strided** convolution and pooling operators may greatly diverge from shift invariance, due to **aliasing** when subsampling high-frequency signals

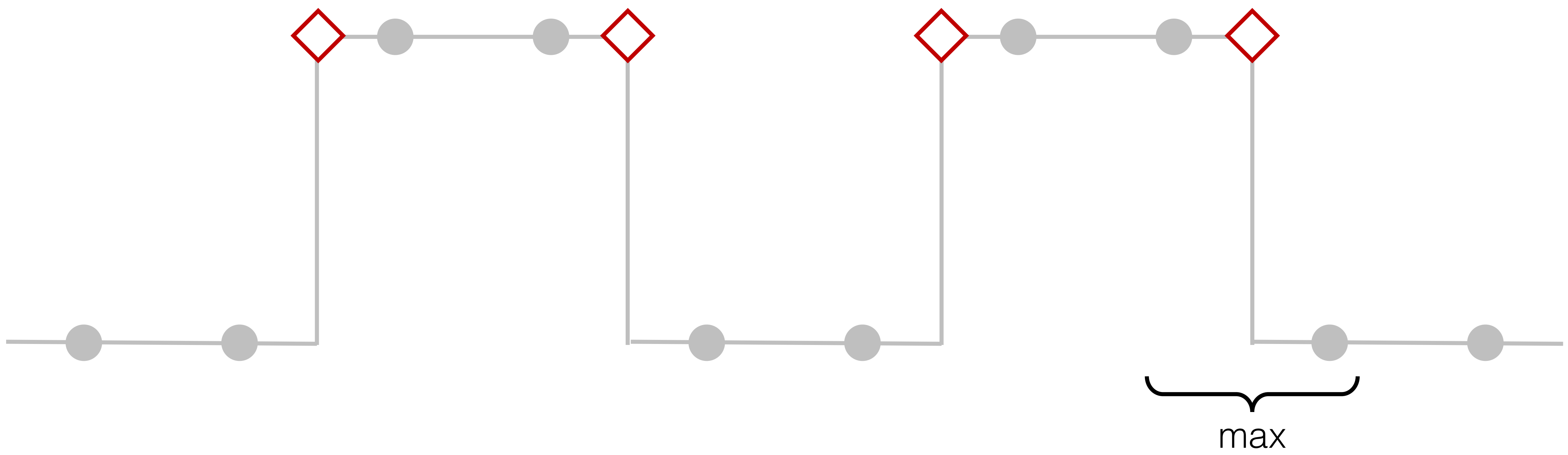


R. Zhang, Making Convolutional Networks Shift-Invariant Again, in International Conference on Machine Learning, 2019.

A. Azulay and Y. Weiss, Why do deep convolutional networks generalize so poorly to small image transformations?, Journal of Machine Learning Research, 2019.

Invariance studies in CNN

- These results do not fully extend to the ***discrete framework***
- **Strided** convolution and pooling operators may greatly diverge from shift invariance, due to **aliasing** when subsampling high-frequency signals

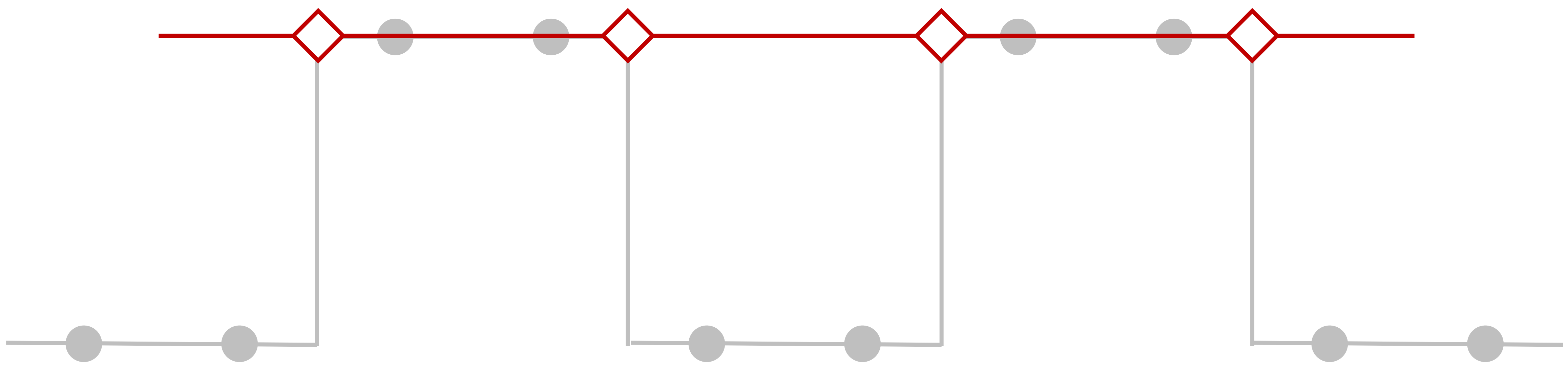


R. Zhang, Making Convolutional Networks Shift-Invariant Again, in International Conference on Machine Learning, 2019.

A. Azulay and Y. Weiss, Why do deep convolutional networks generalize so poorly to small image transformations?, Journal of Machine Learning Research, 2019.

Invariance studies in CNN

- These results do not fully extend to the ***discrete framework***
- **Strided** convolution and pooling operators may greatly diverge from shift invariance, due to **aliasing** when subsampling high-frequency signals

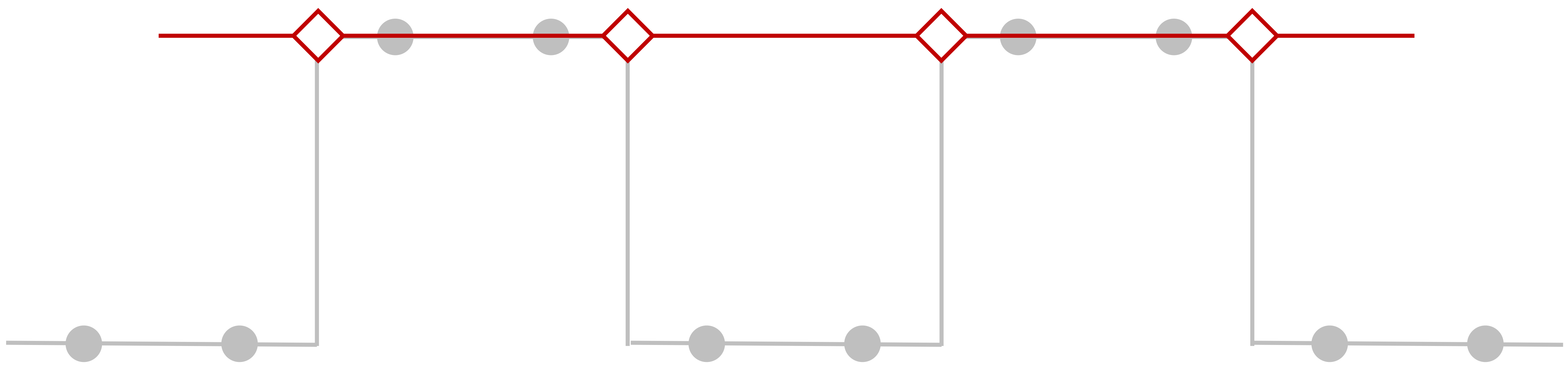


R. Zhang, Making Convolutional Networks Shift-Invariant Again, in International Conference on Machine Learning, 2019.

A. Azulay and Y. Weiss, Why do deep convolutional networks generalize so poorly to small image transformations?, Journal of Machine Learning Research, 2019.

Invariance studies in CNN

- These results do not fully extend to the ***discrete framework***
- **Strided** convolution and pooling operators may greatly diverge from shift invariance, due to **aliasing** when subsampling high-frequency signals

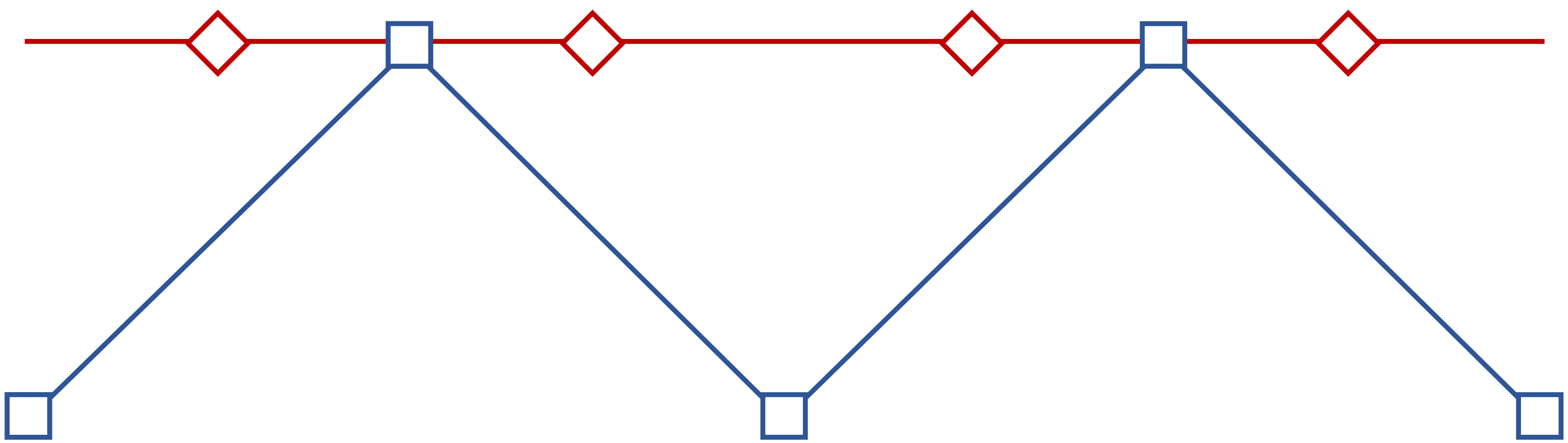


R. Zhang, Making Convolutional Networks Shift-Invariant Again, in International Conference on Machine Learning, 2019.

A. Azulay and Y. Weiss, Why do deep convolutional networks generalize so poorly to small image transformations?, Journal of Machine Learning Research, 2019.

Invariance studies in CNN

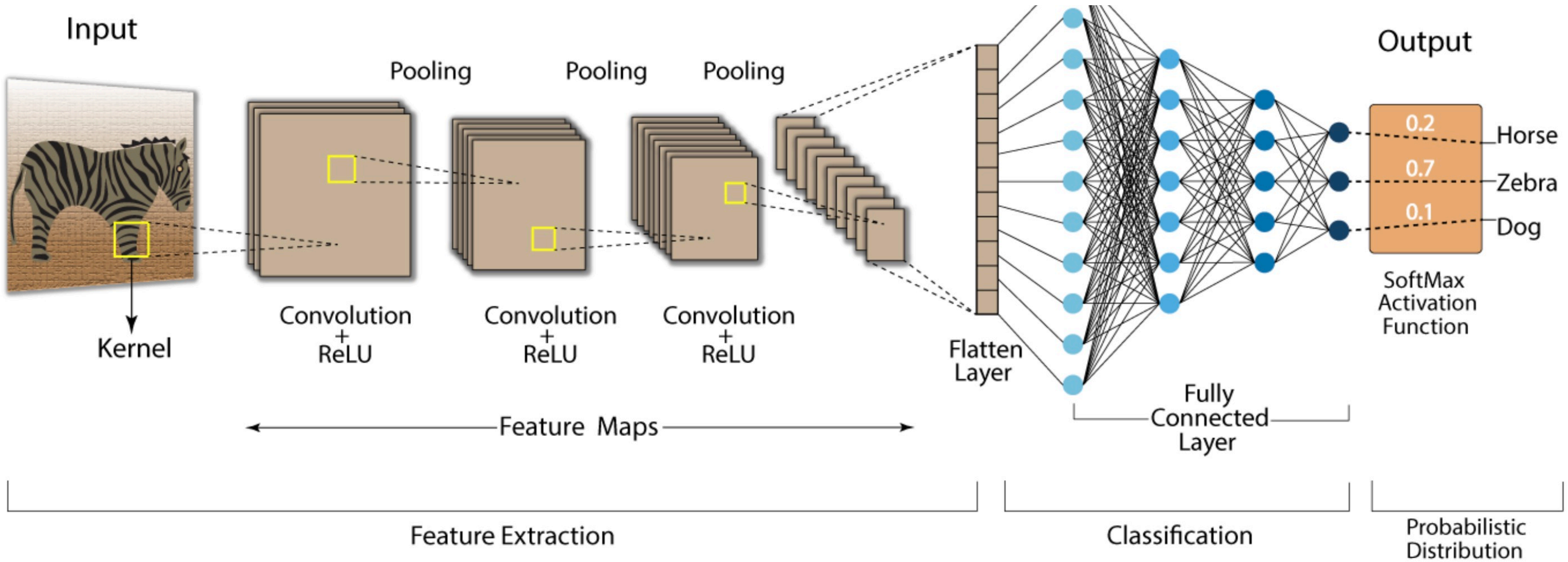
- These results do not fully extend to the ***discrete framework***
- **Strided** convolution and pooling operators may greatly diverge from shift invariance, due to **aliasing** when subsampling high-frequency signals



R. Zhang, Making Convolutional Networks Shift-Invariant Again, in International Conference on Machine Learning, 2019.

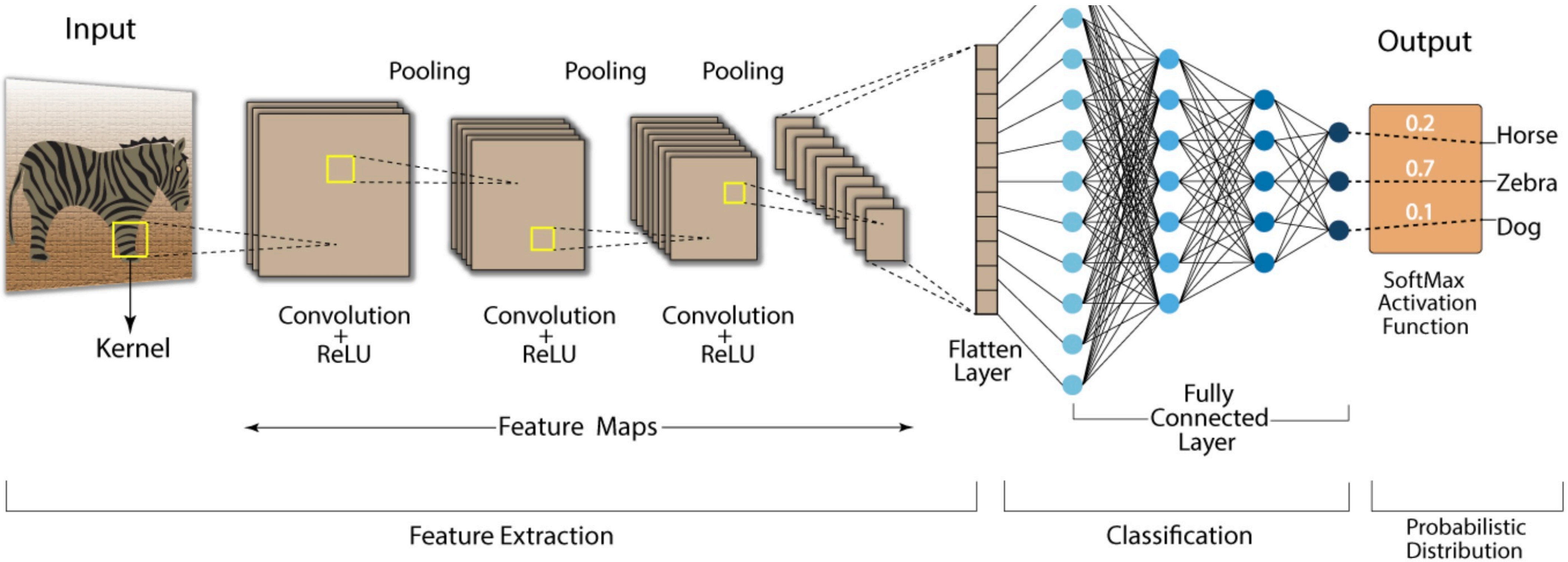
A. Azulay and Y. Weiss, Why do deep convolutional networks generalize so poorly to small image transformations?, Journal of Machine Learning Research, 2019.

Aliasing breaks shift-invariance

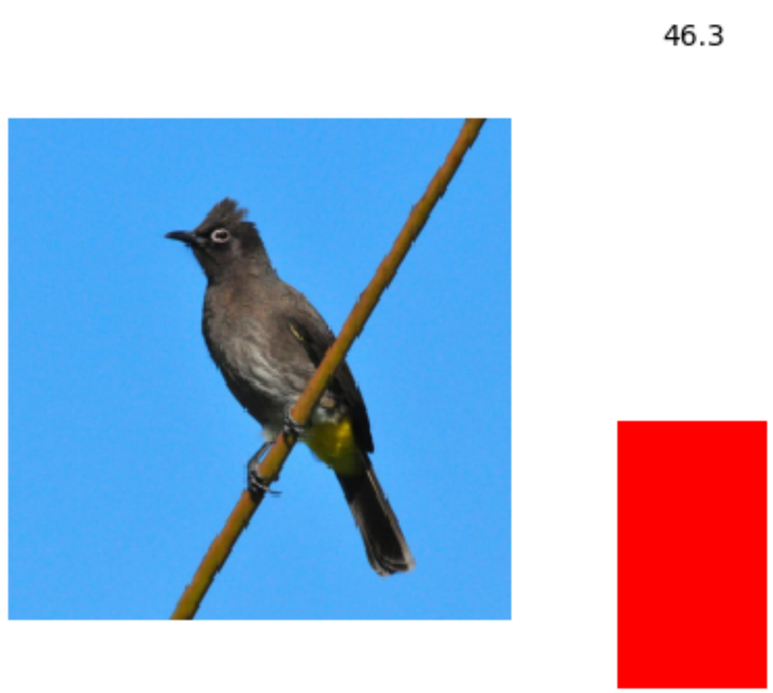


Source : https://developersbreach.com/convolution-neural-network-deep-learning/

Aliasing breaks shift-invariance



Source : https://developersbreach.com/convolution-neural-network-deep-learning/



Blind spots in the shift-invariance studies

Blind spots in the shift-invariance studies

- The **effect of the max pooling operator on network stability** under small input shifts has not been investigated, particularly when used in combination with Gabor-like convolutions.

Blind spots in the shift-invariance studies

- The **effect of the max pooling operator on network stability** under small input shifts has not been investigated, particularly when used in combination with Gabor-like convolutions.

- In the discrete case, the presence of subsampled convolutions with oriented band-pass filters can lead to aliasing artifacts. To our knowledge, **the literature lacks theoretical studies that take these aliasing effects into account**.

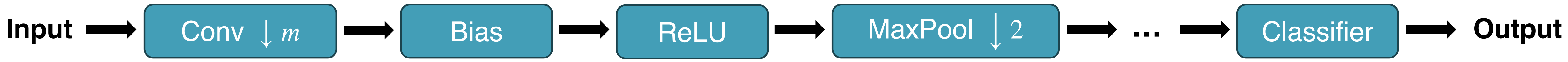
Blind spots in the shift-invariance studies

- The **effect of the max pooling operator on network stability** under small input shifts has not been investigated, particularly when used in combination with Gabor-like convolutions.

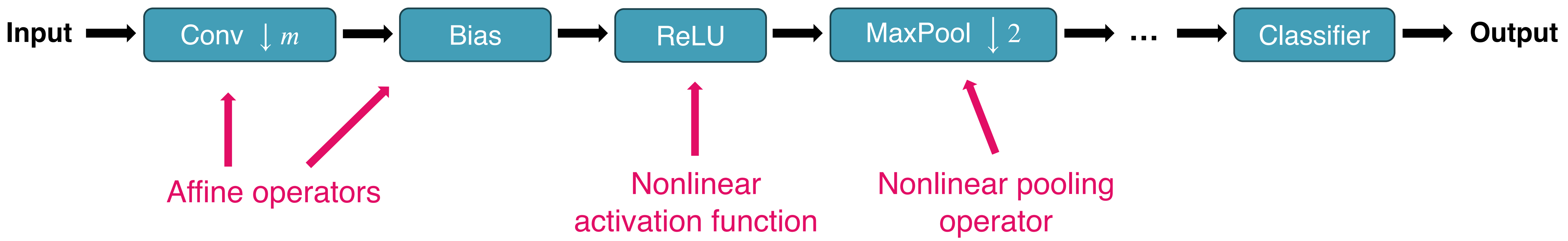
- In the discrete case, the presence of subsampled convolutions with oriented band-pass filters can lead to aliasing artifacts. To our knowledge, **the literature lacks theoretical studies that take these aliasing effects into account**.

- Although extensive studies have been conducted on complex-valued convolutions followed by modulus, **a link is missing to extend these results to standard CNNs**, which implement real-valued convolutions and spatial pooling operators.

Focus on the first layer

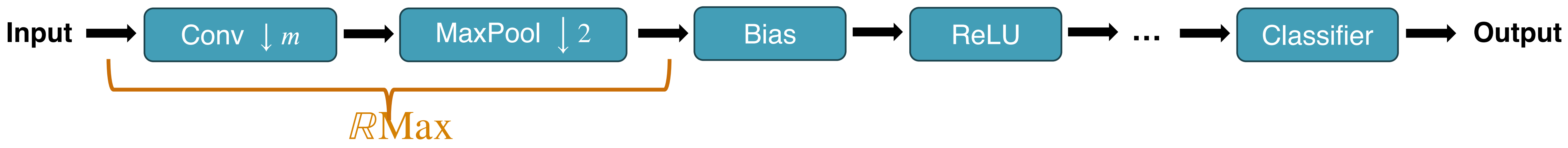


Focus on the first layer

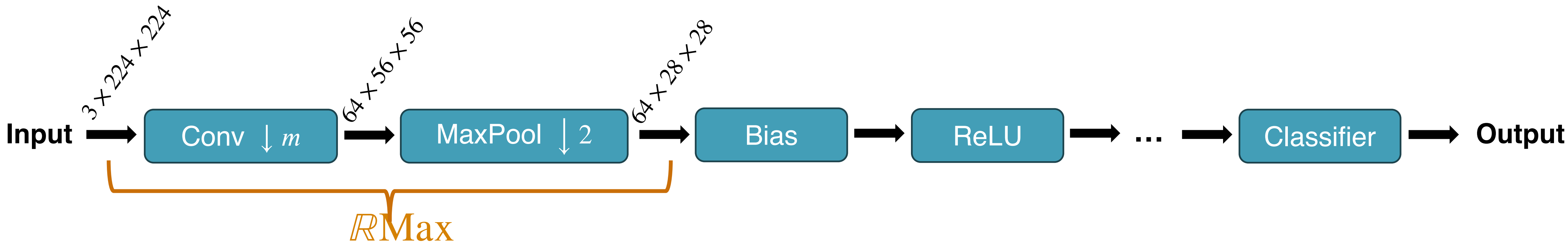


Focus on the first layer

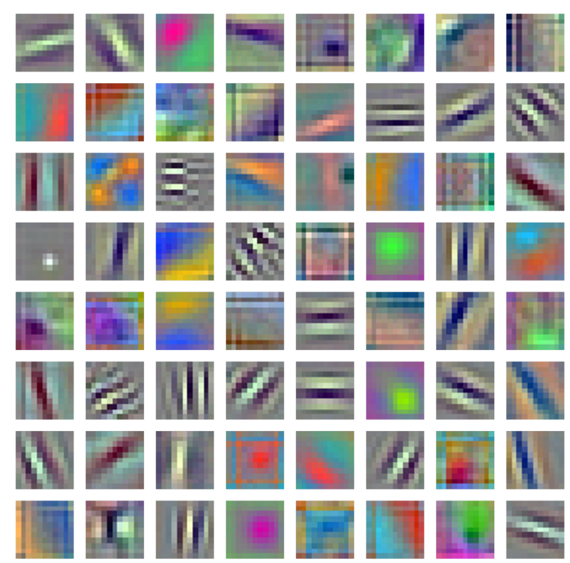
Focus on the first layer



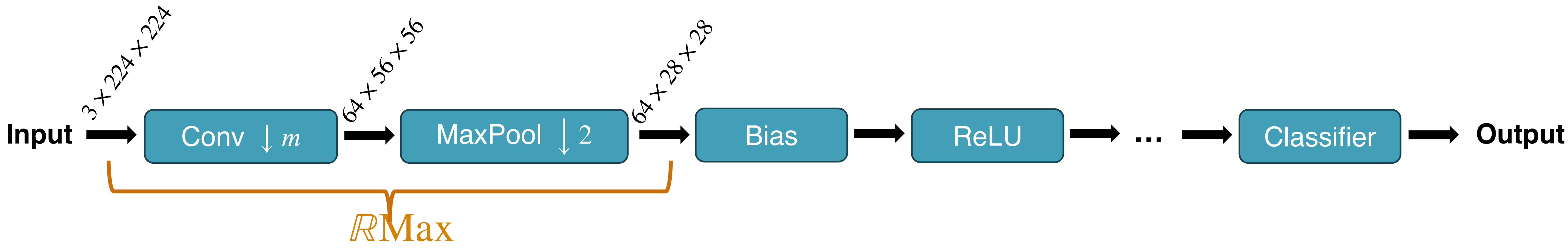
Focus on the first layer



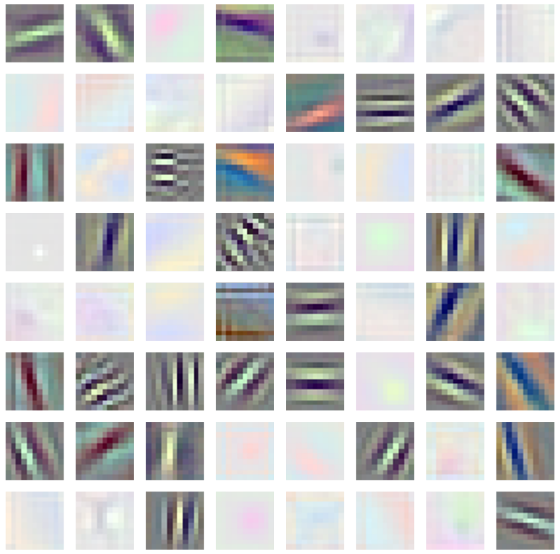
Example: AlexNet
(2012)



Focus on the first layer

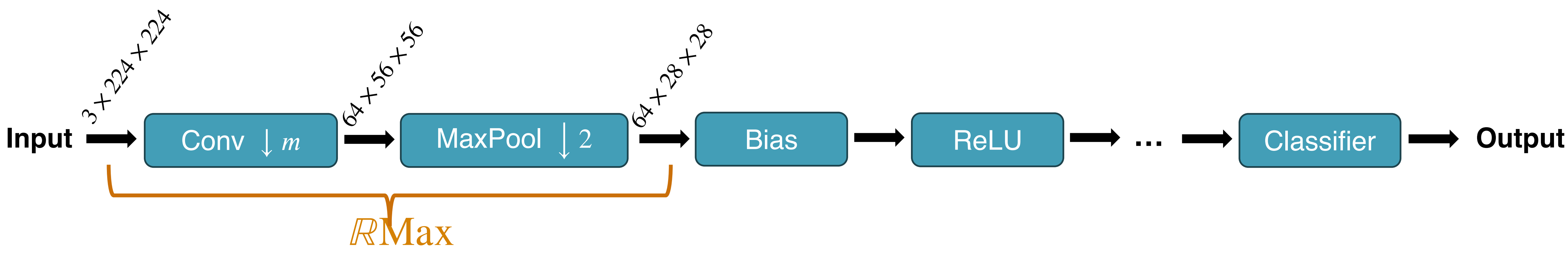


Band-pass **"Gabor-like"** filters



Example: AlexNet (2012)

Focus on the first layer



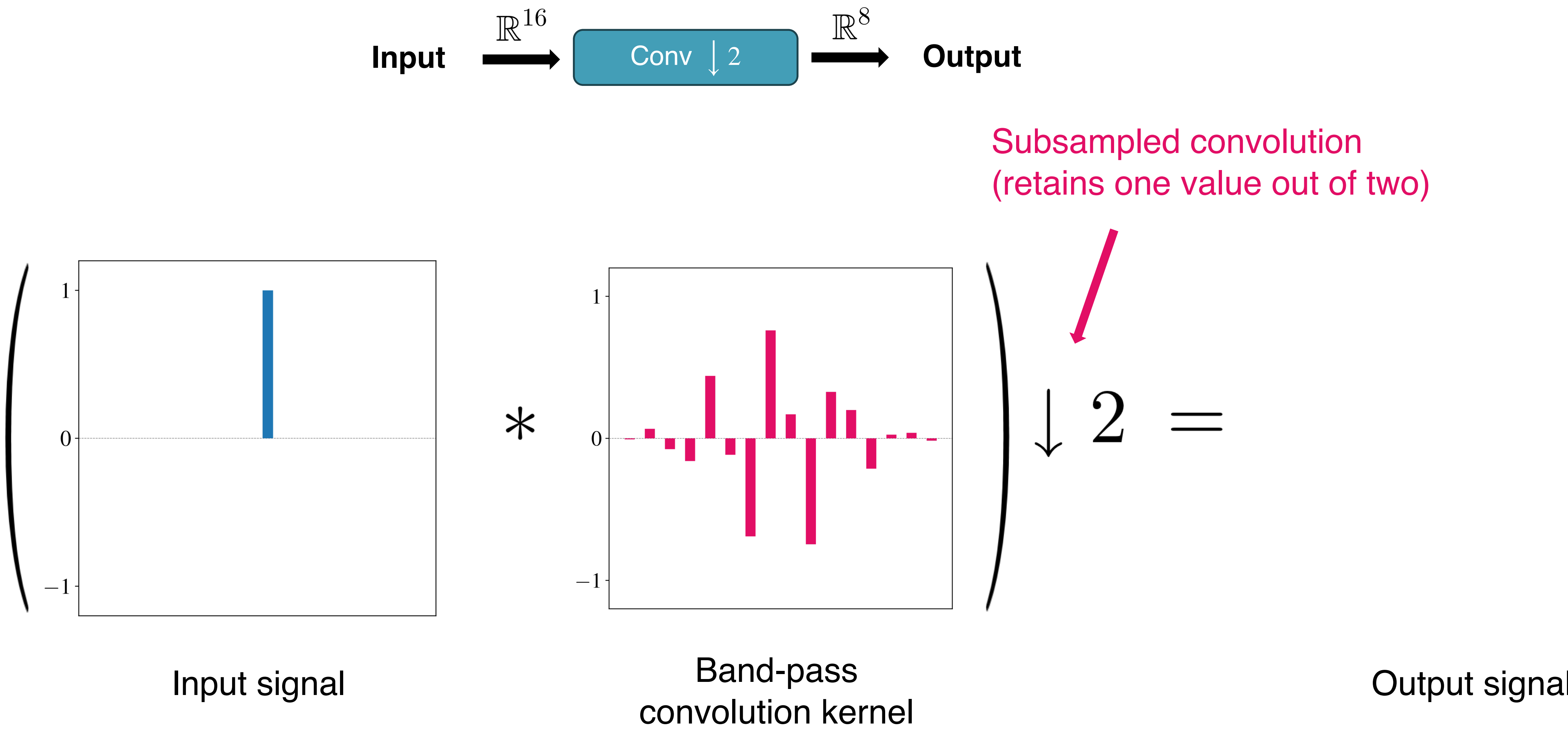
Band-pass **"Gabor-like"** filters

Example: AlexNet
(2012)

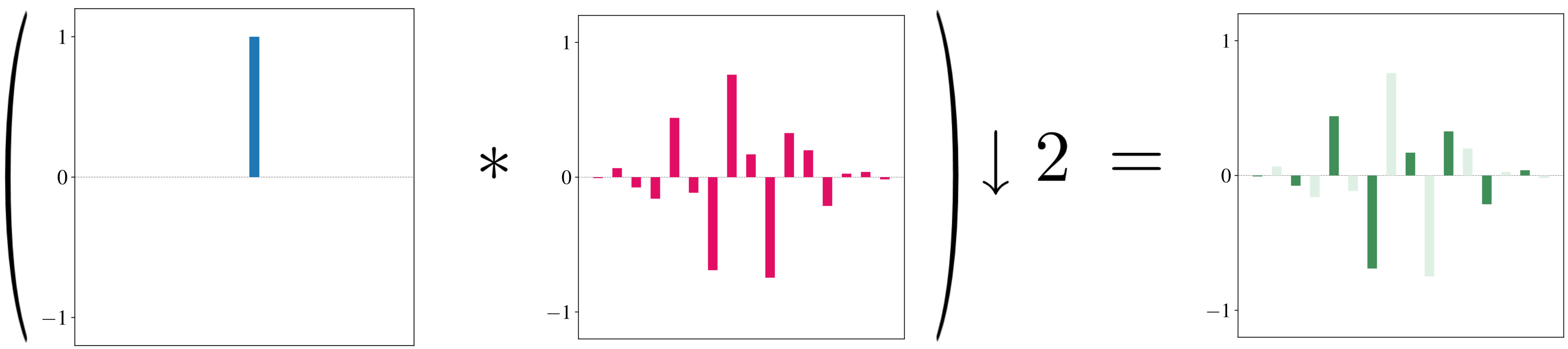
Rai, Mehang, and Pablo Rivas. "A review of convolutional neural networks and Gabor filters in object recognition." *2020 International Conference on Computational Science and Computational Intelligence (CSCI)*. IEEE, 2020.

Yosinski J, Clune J, Bengio Y, and Lipson H. How transferable are features in deep neural networks? In Advances in Neural Information Processing Systems 27 (NIPS '14), NIPS Foundation, 2014.

Subsampled convolutions, a real problem!



Subsampled convolutions, a real problem!

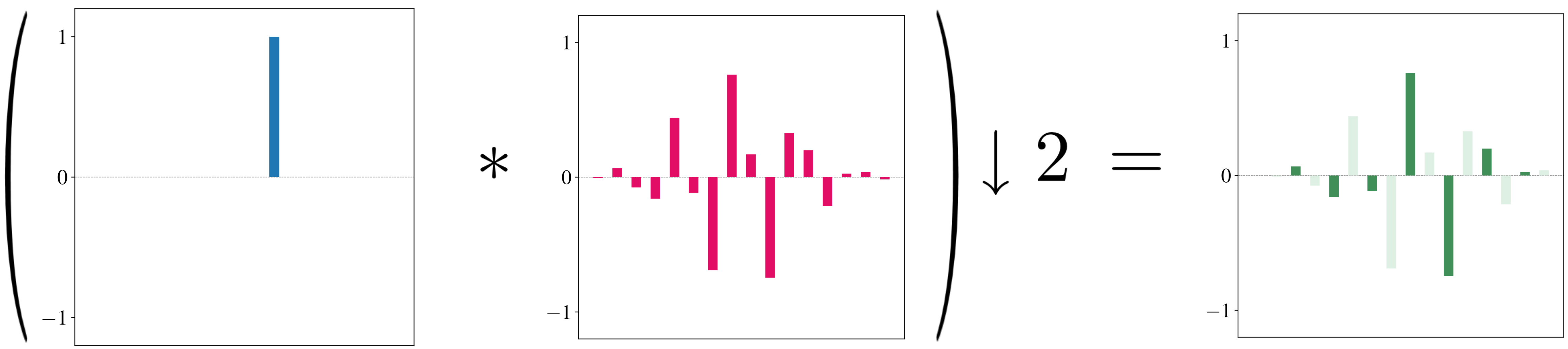


Input signal

Band-pass convolution kernel

Output signal

Subsampled convolutions, a real problem!



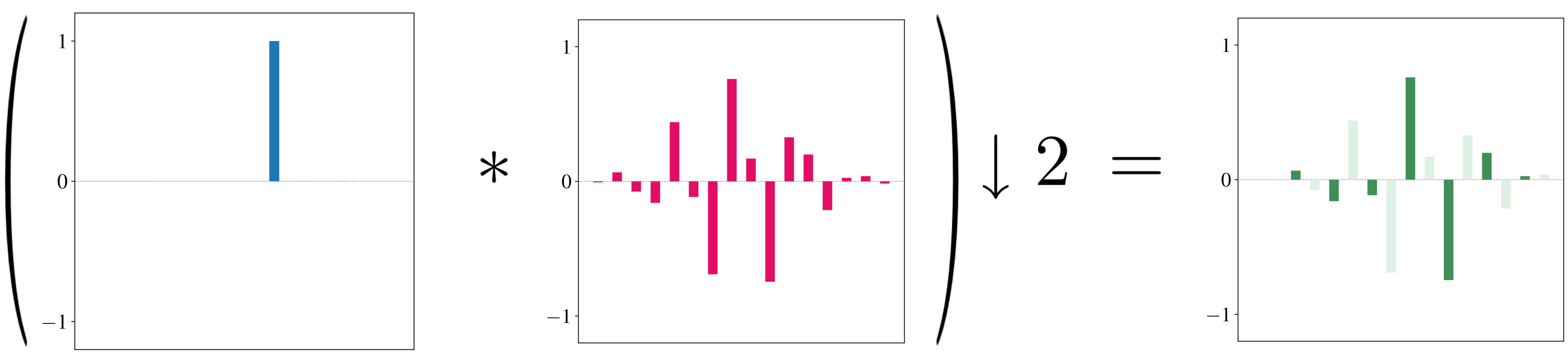
Input signal

Band-pass
convolution kernel

Output signal

Subsampled convolutions, a real problem!

Subsampling a high-frequency signal
→ **Aliasing effect**
→ **Instability to small input shifts**

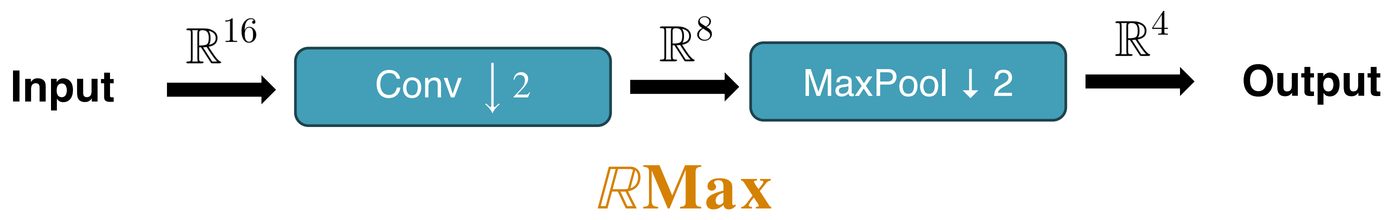


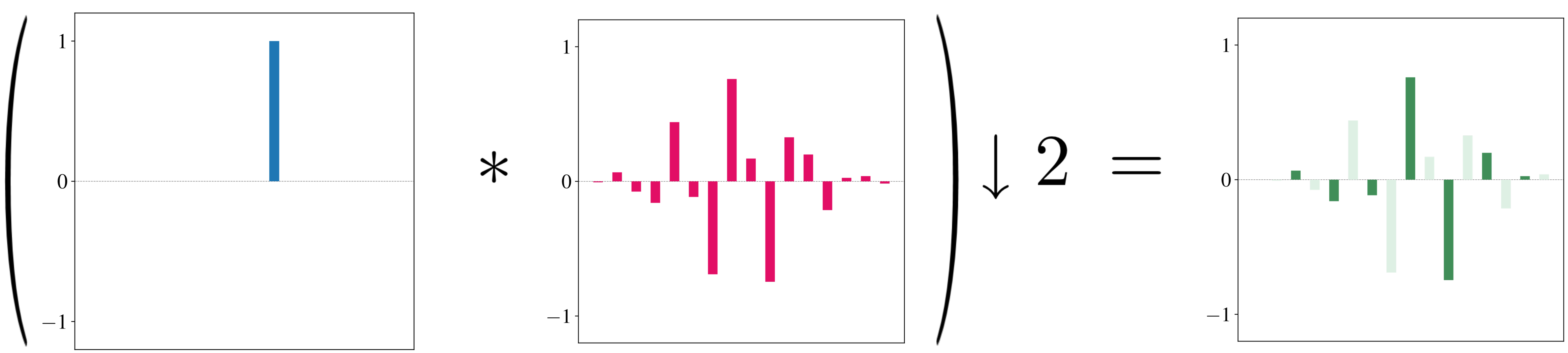
Input signal

Band-pass convolution kernel

Output signal

Subsampled convolutions, a real problem!



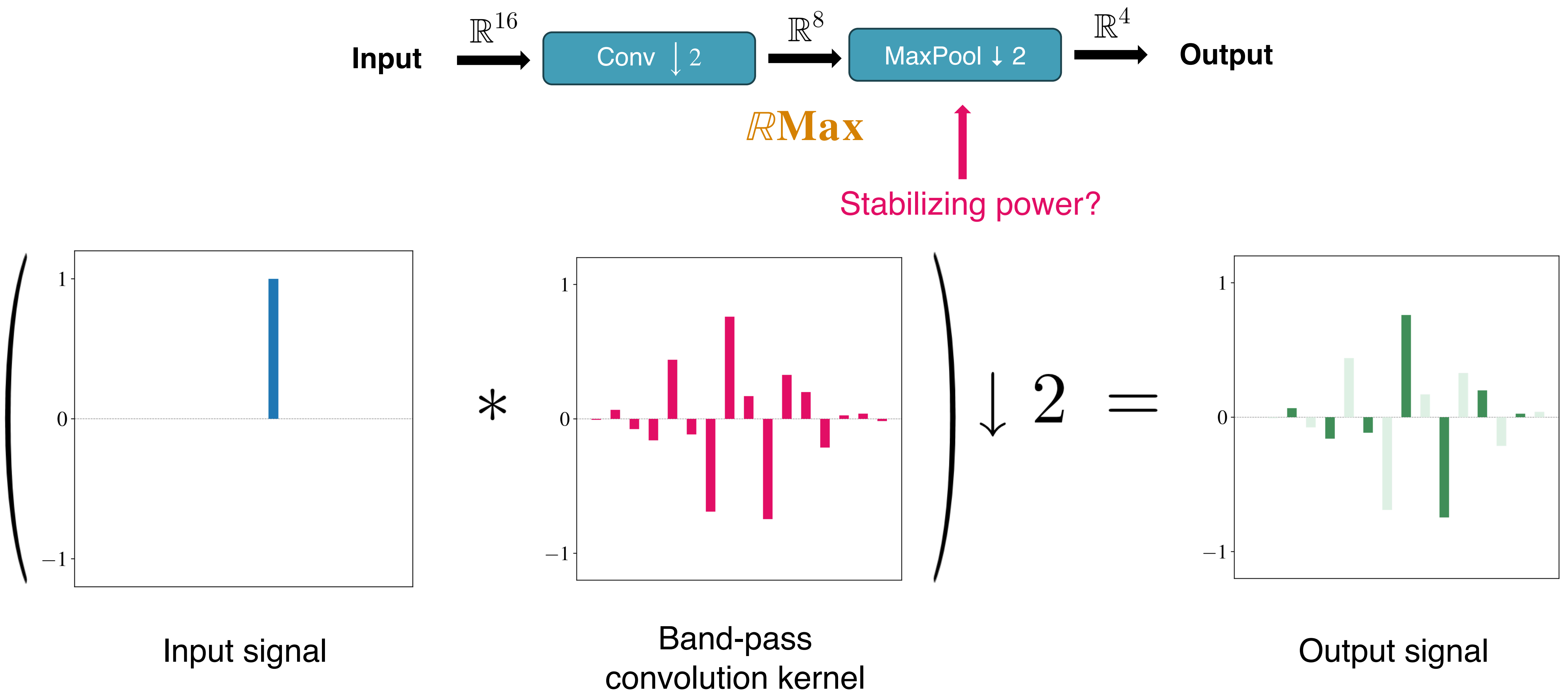


Input signal

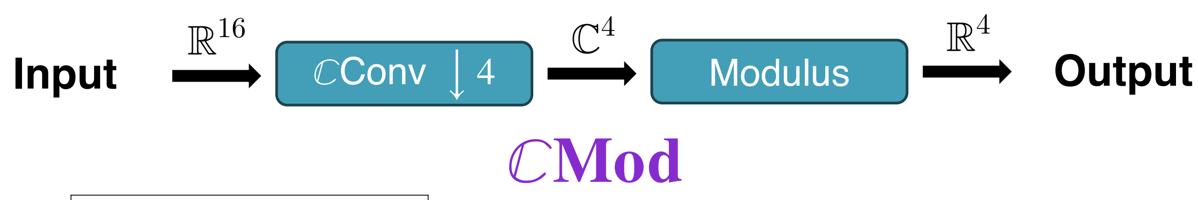
Band-pass
convolution kernel

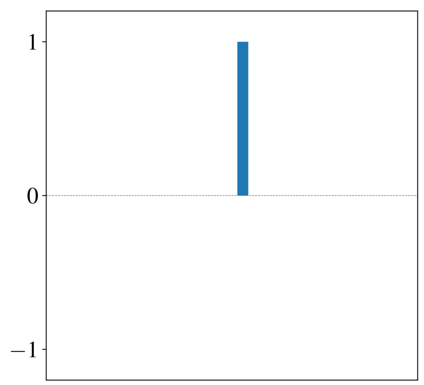
Output signal

Subsampled convolutions, a real problem!



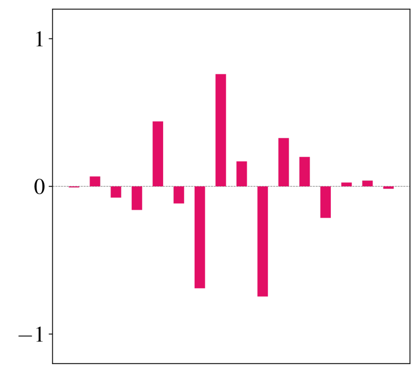
Complex-valued convolutions at rescue



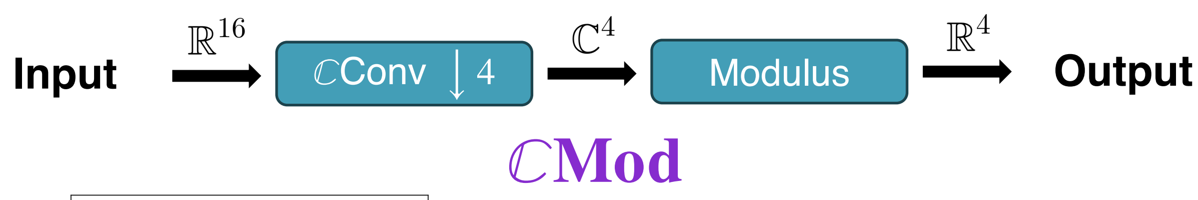


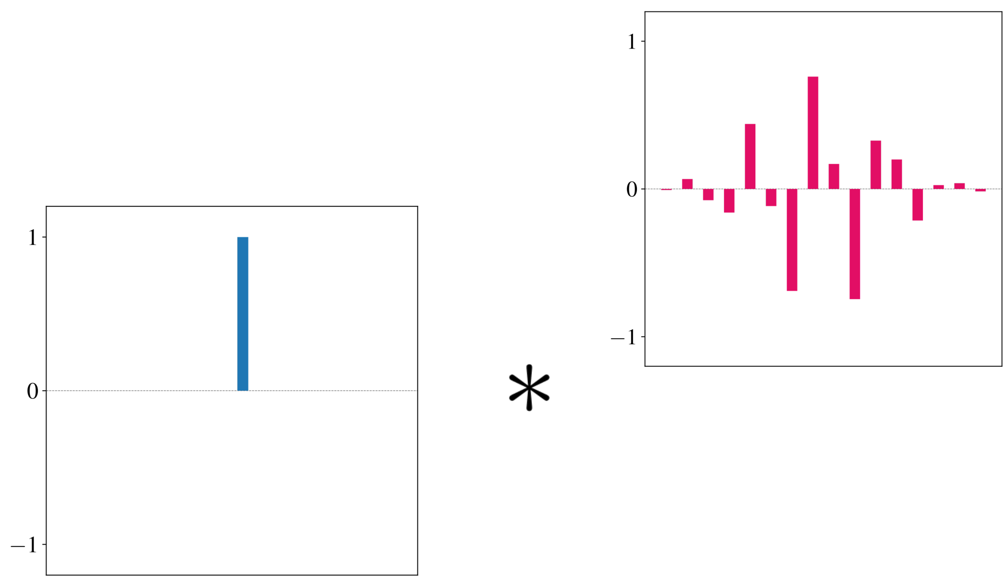
Input signal

$*$

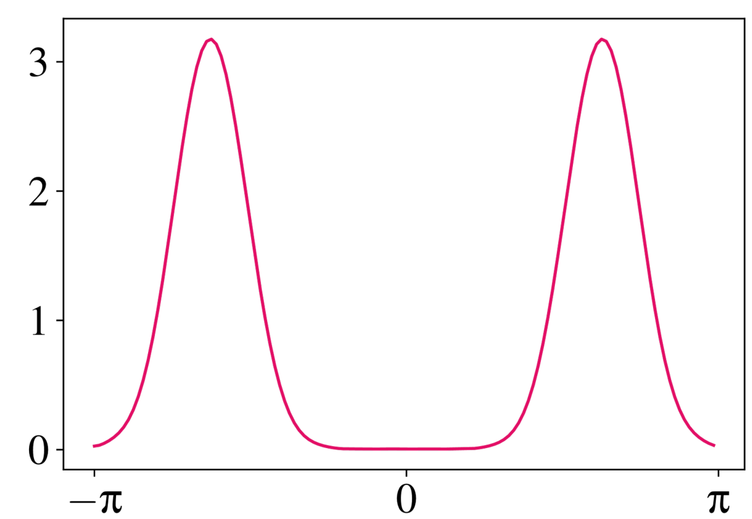


Complex-valued convolutions at rescue



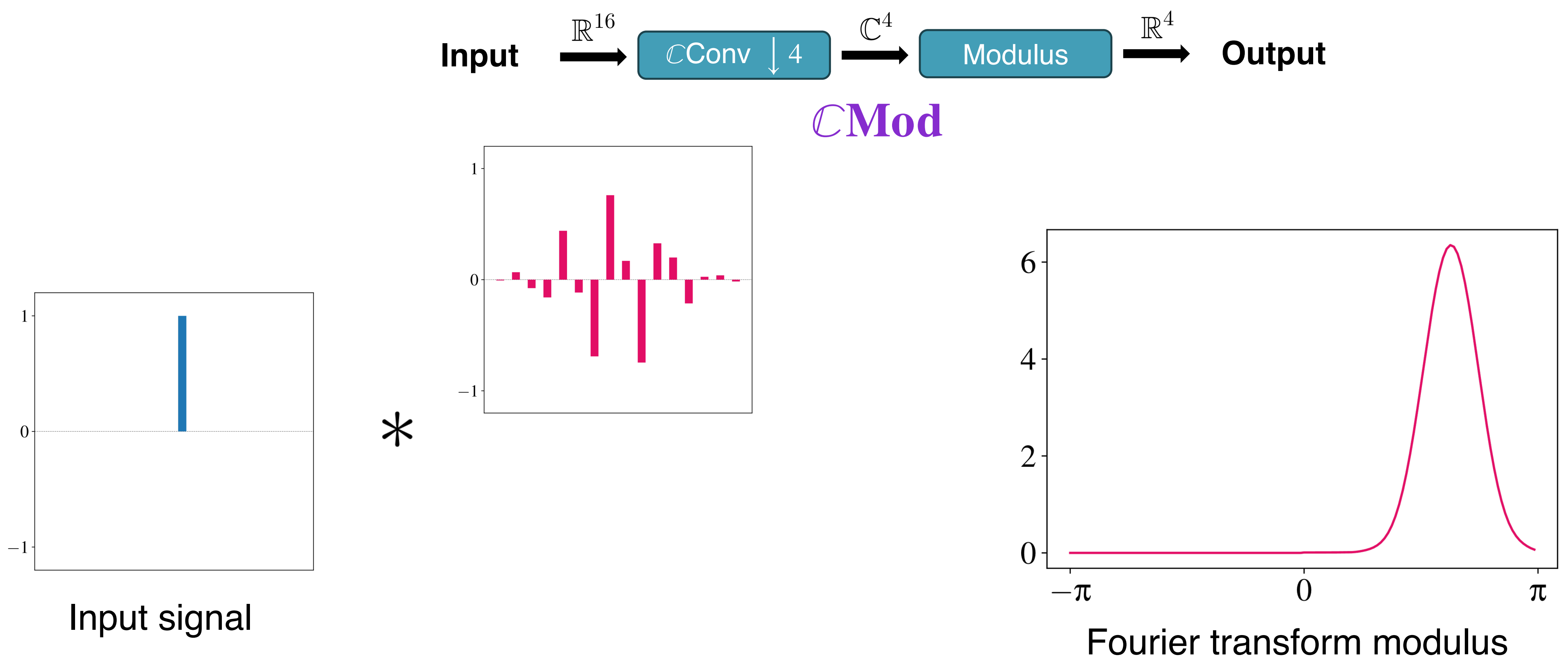


Input signal

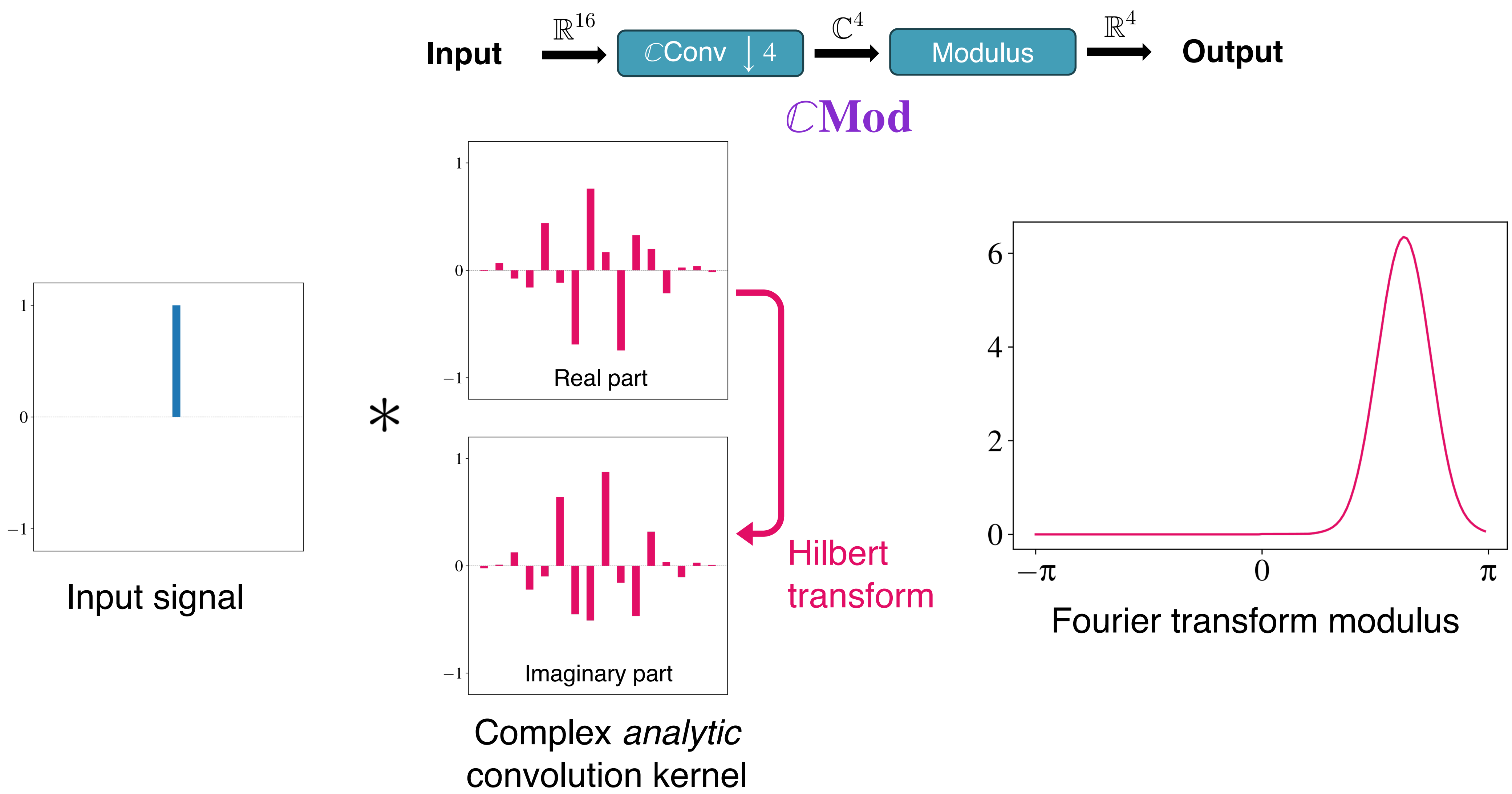


Fourier transform modulus

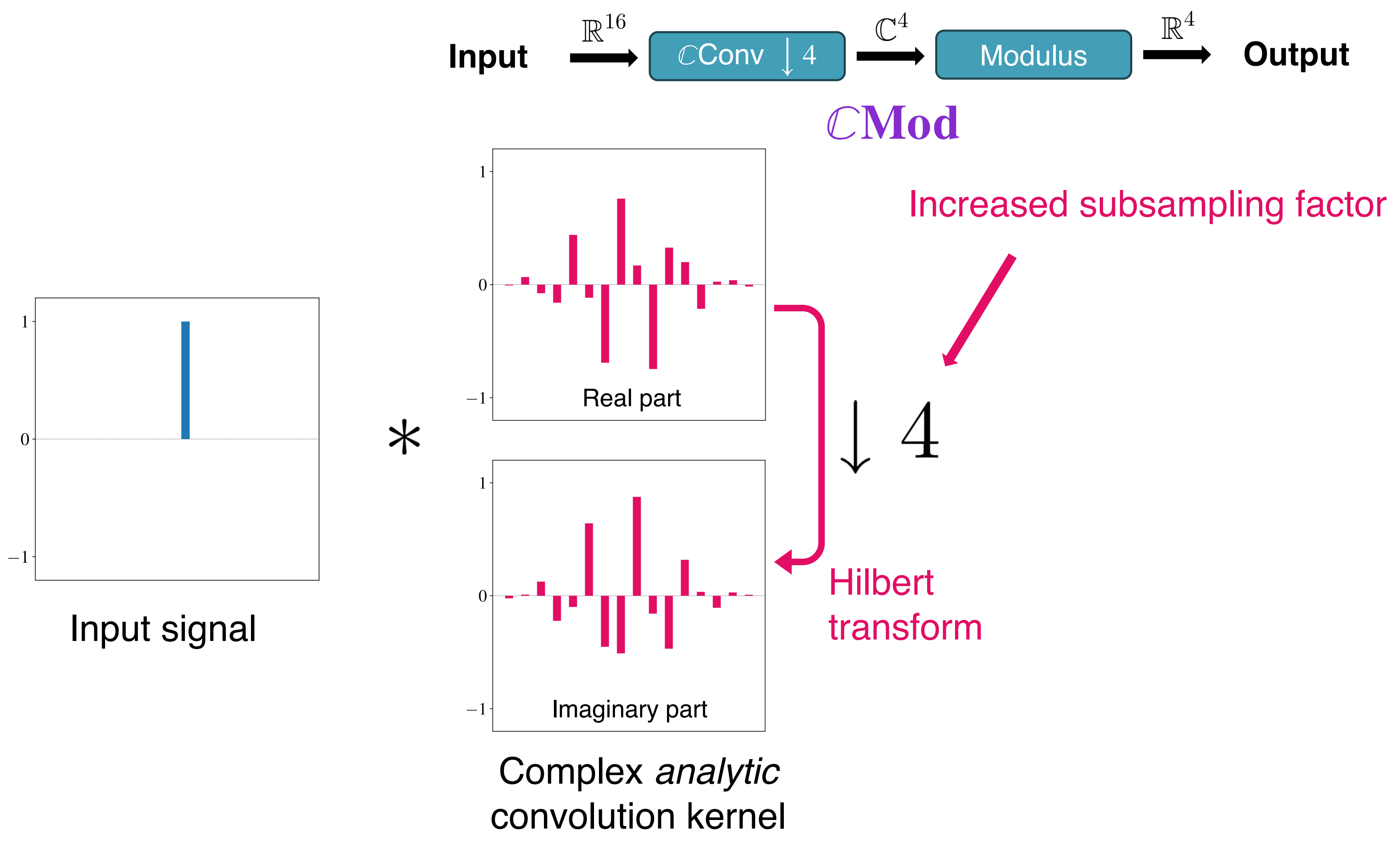
Complex-valued convolutions at rescue



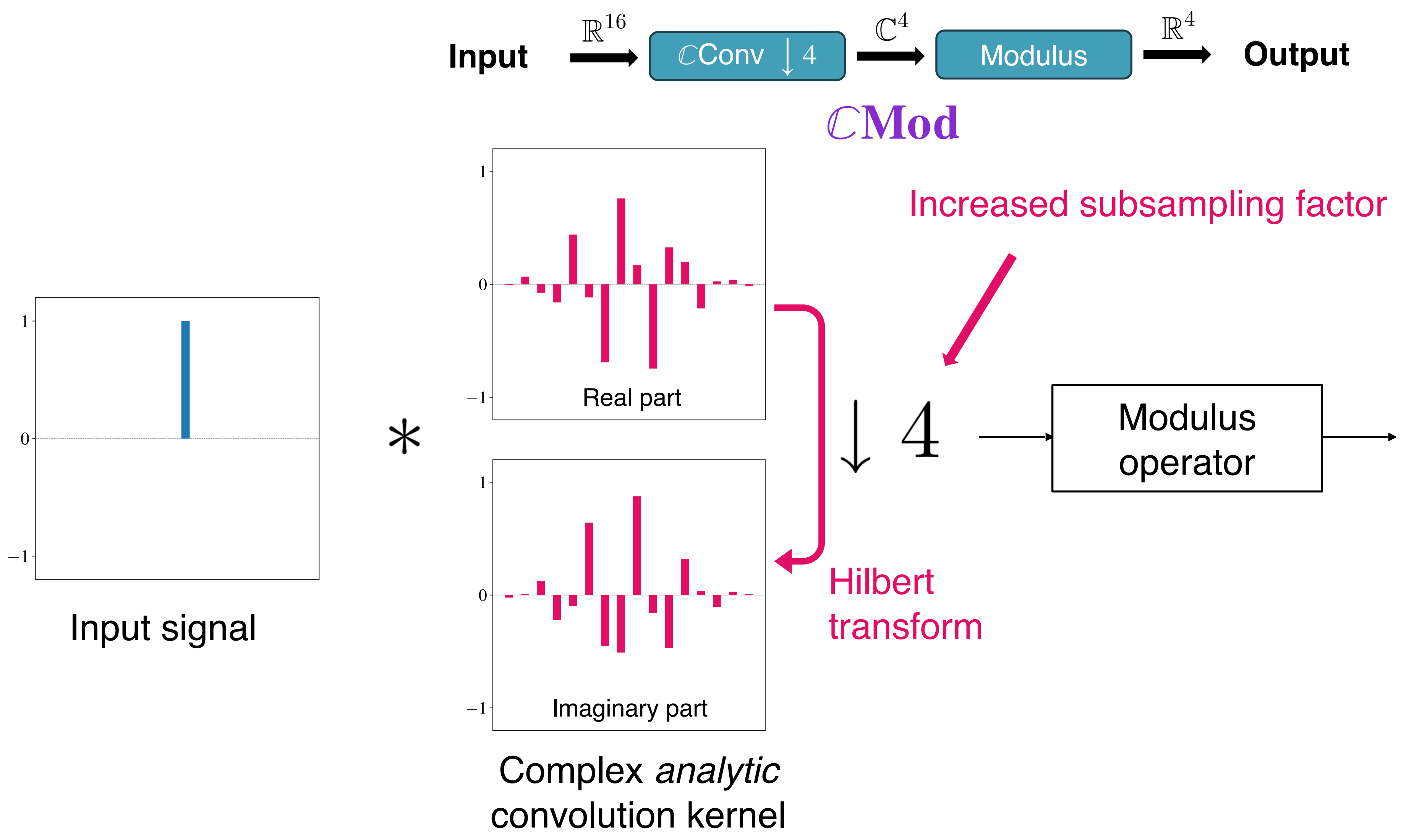
Complex-valued convolutions at rescue



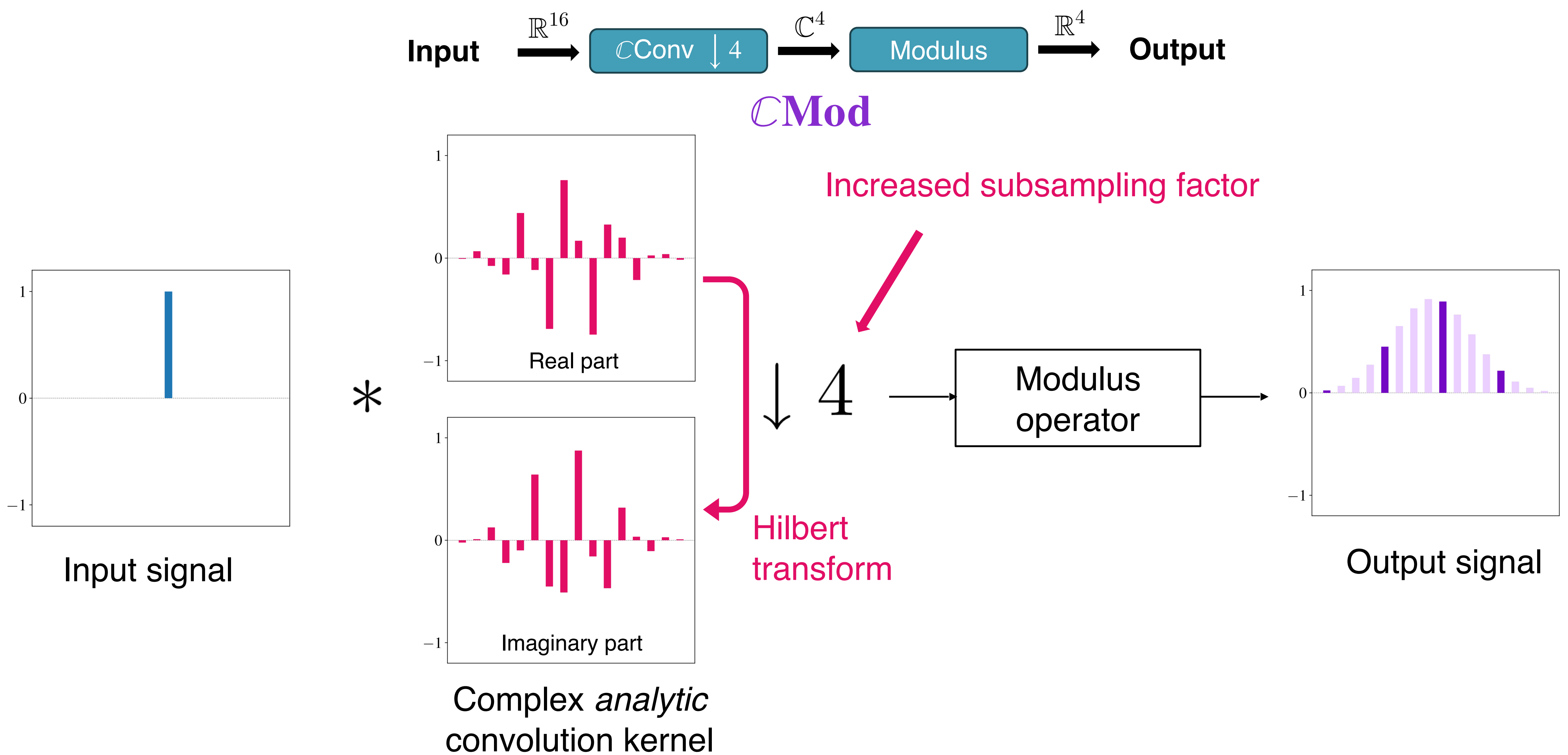
Complex-valued convolutions at rescue



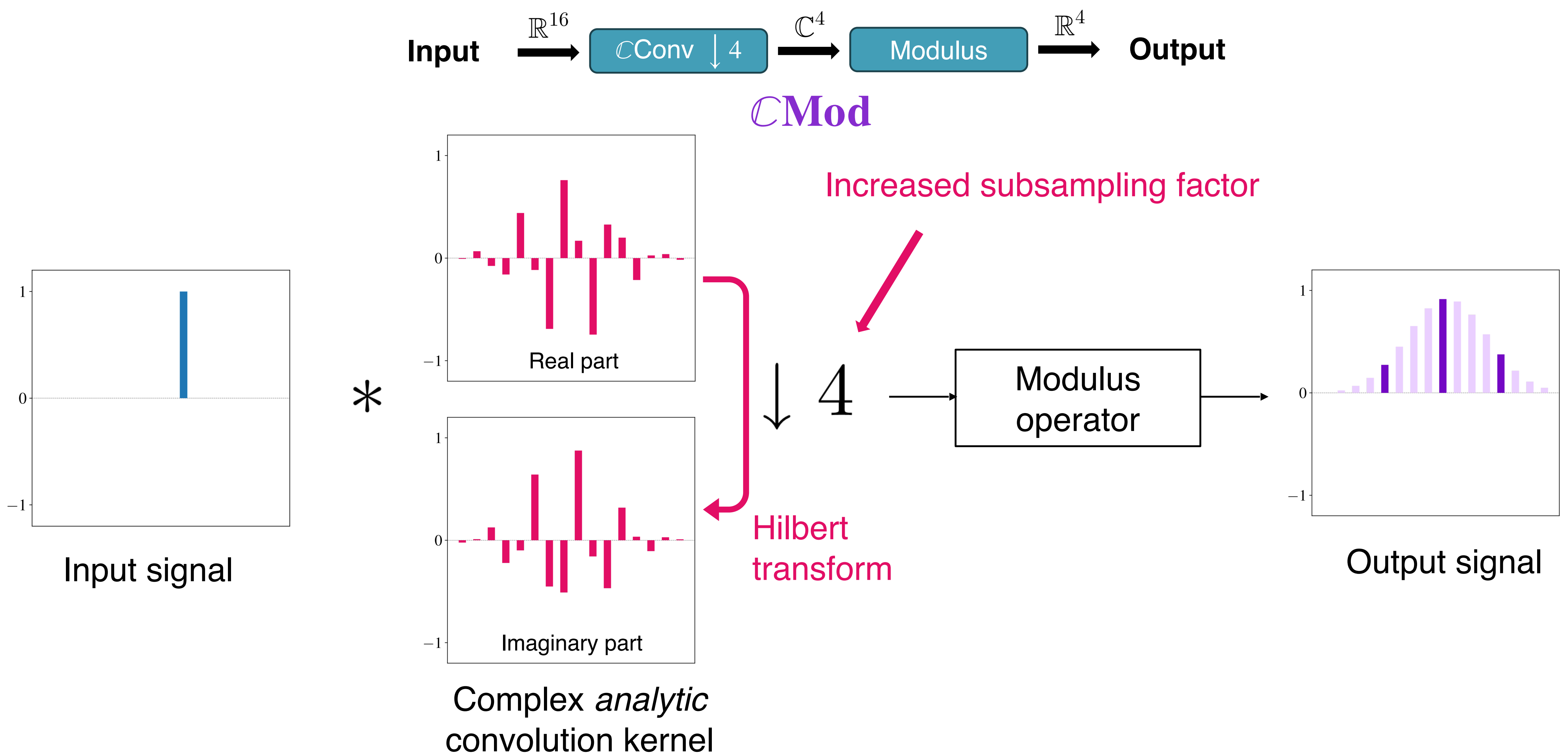
Complex-valued convolutions at rescue



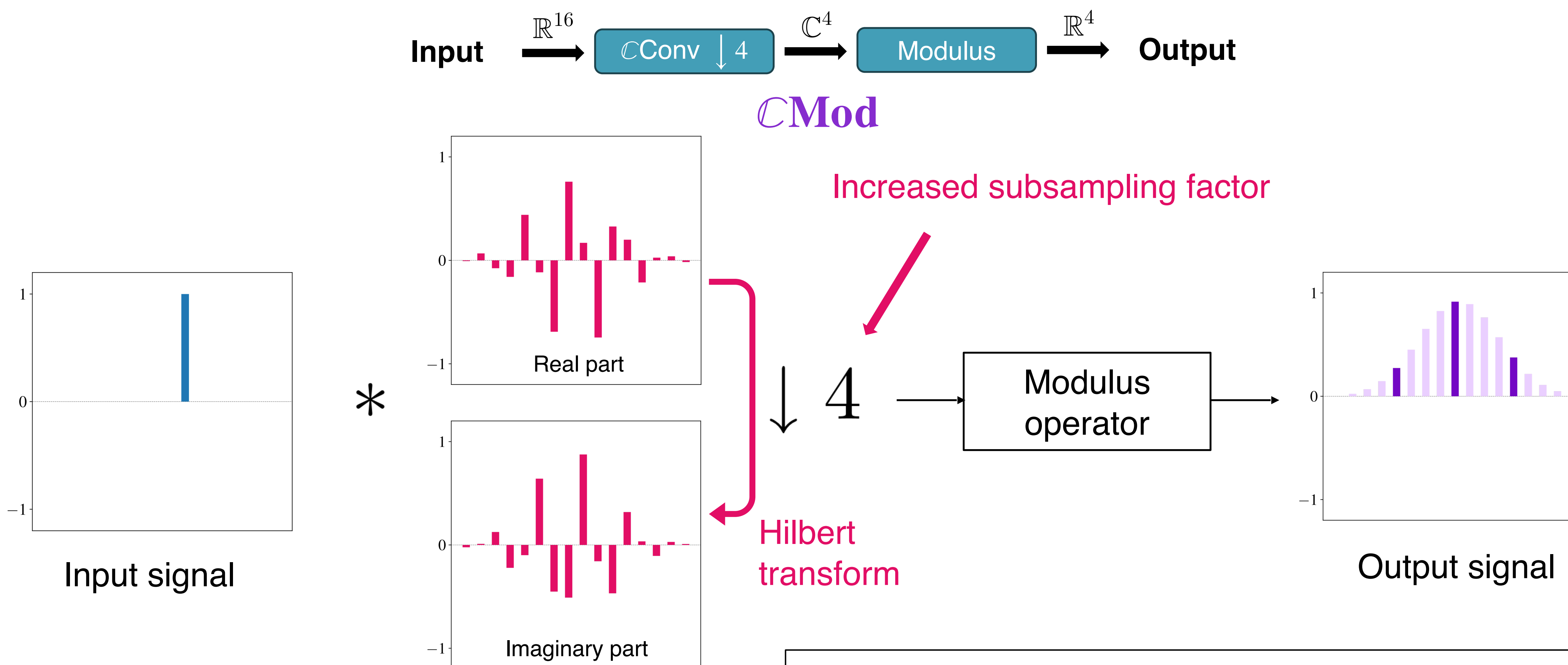
Complex-valued convolutions at rescue



Complex-valued convolutions at rescue



Complex-valued convolutions at rescue



Complex *analytic* convolution kernel

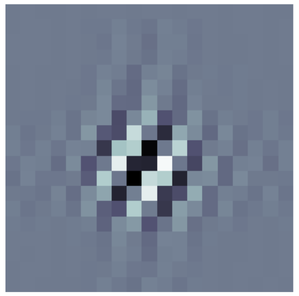
Objective: establish conditions under which $\mathbb{R}\text{Max} \approx \mathbb{C}\text{Mod}$.

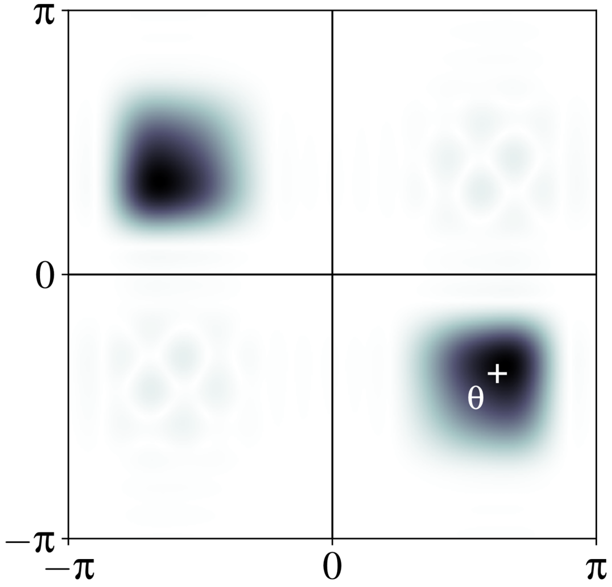
Fundamental hypothesis on filter

- Band-pass, oriented and **analytic** *Gabor-like* filters W

Fundamental hypothesis on filter

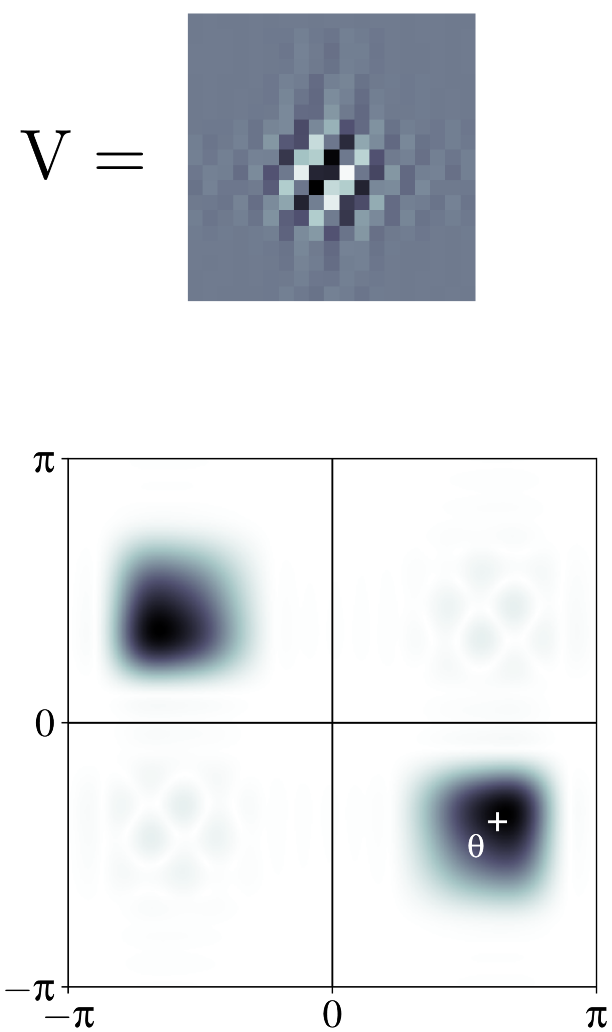
- Band-pass, oriented and **analytic** *Gabor-like* filters W

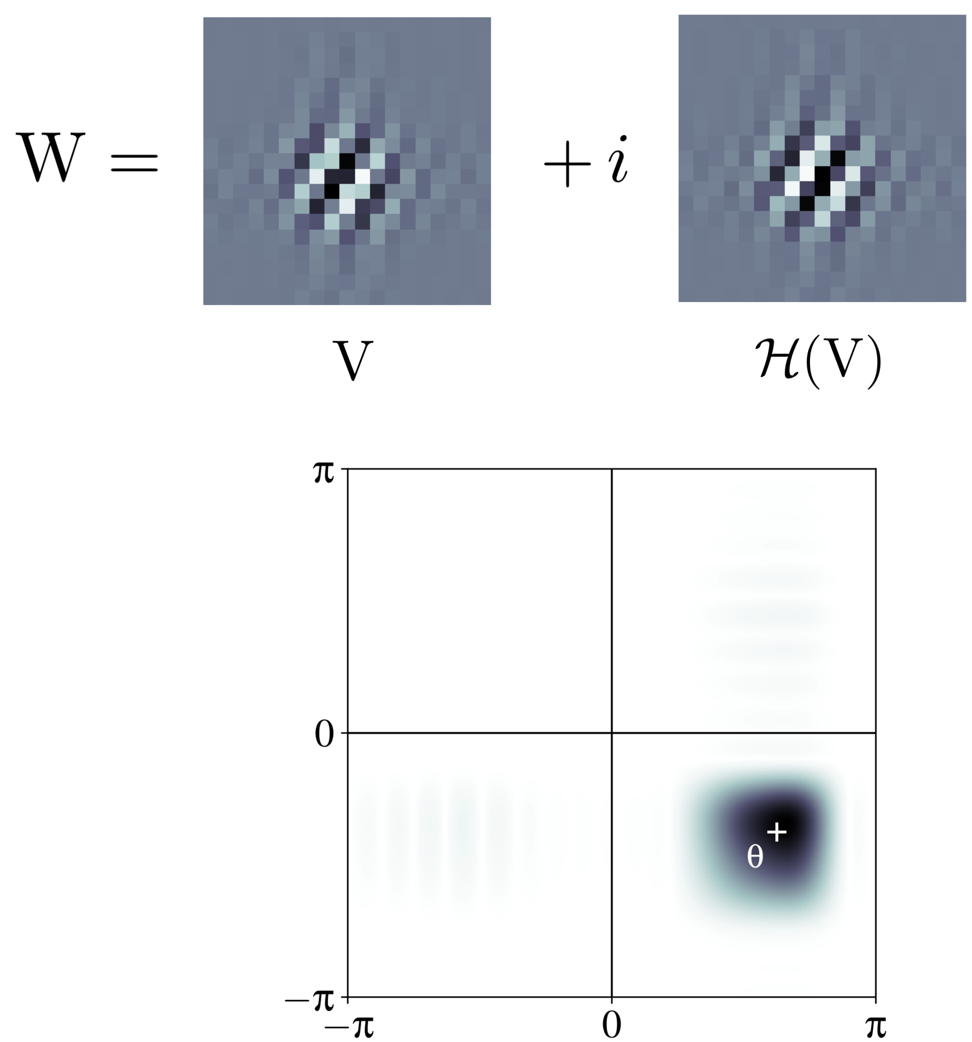
$\mathrm{V} =$ 



Fundamental hypothesis on filter

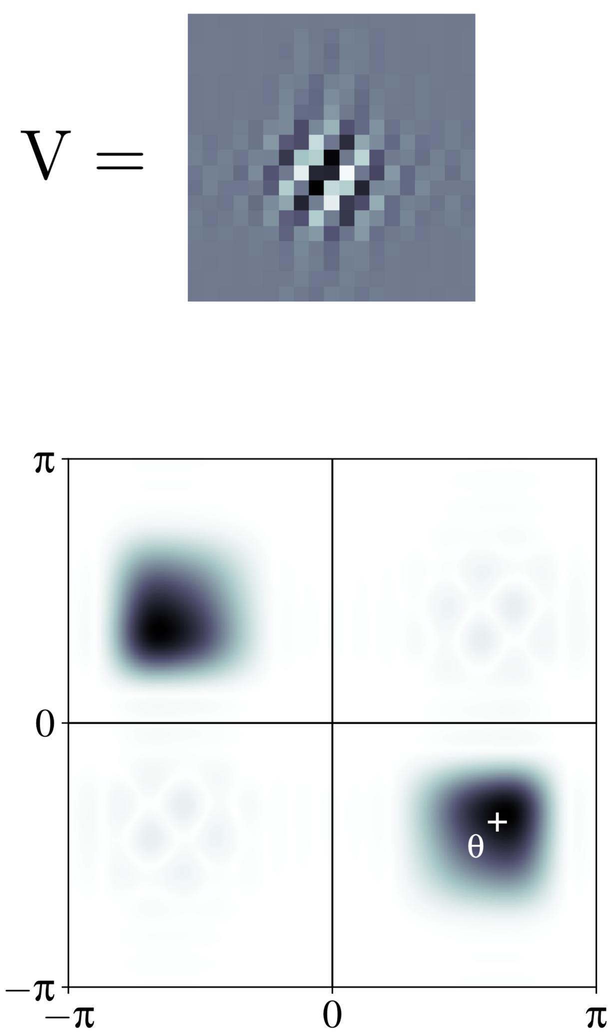
- Band-pass, oriented and **analytic** *Gabor-like* filters W

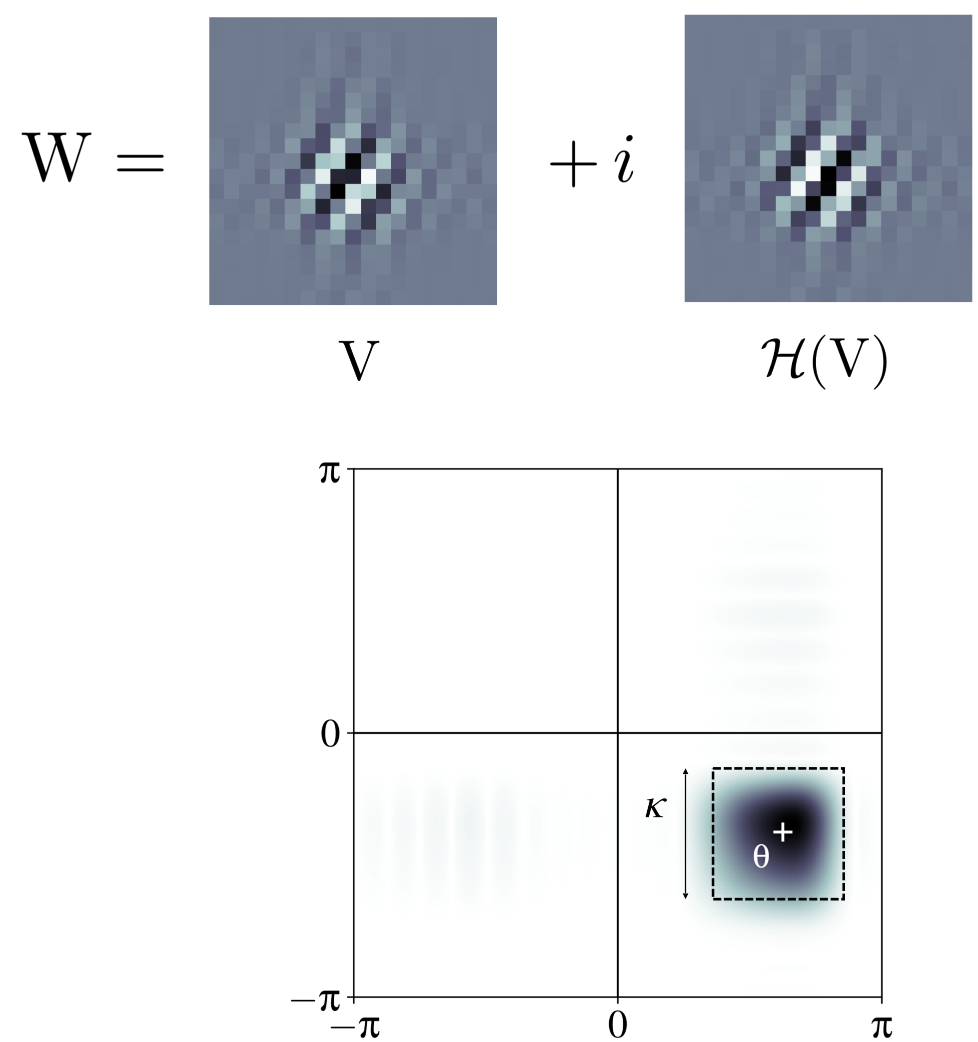




Fundamental hypothesis on filter

- Band-pass, oriented and **analytic** *Gabor-like* filters W

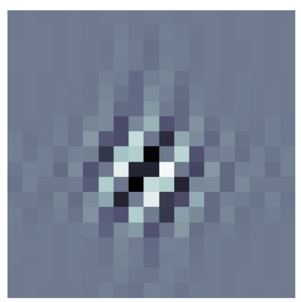


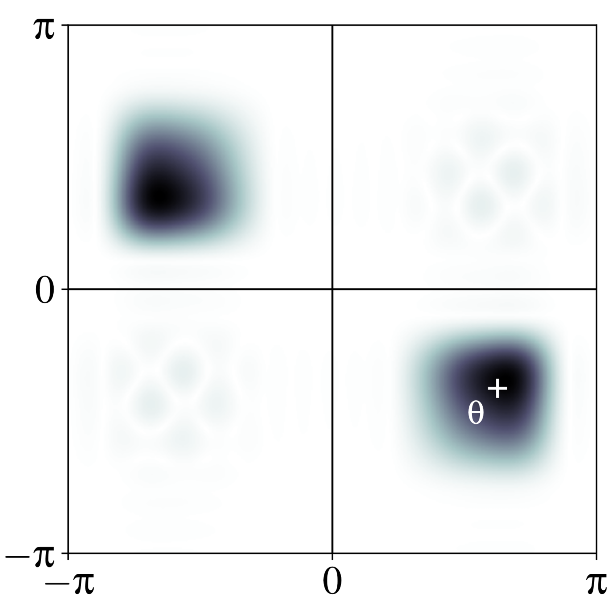


Fundamental hypothesis on filter

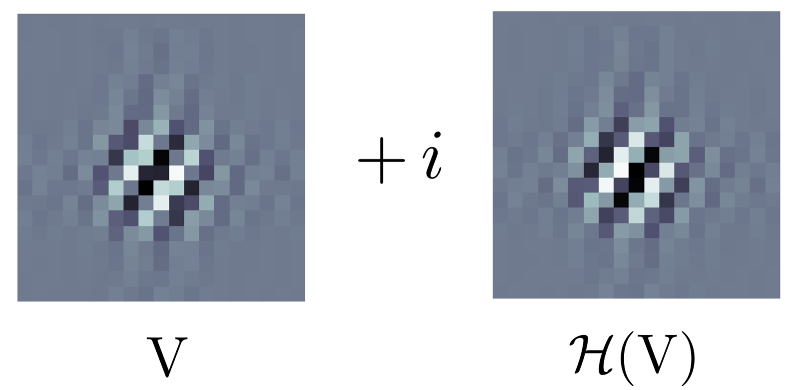
- Band-pass, oriented and **analytic** *Gabor-like* filters W

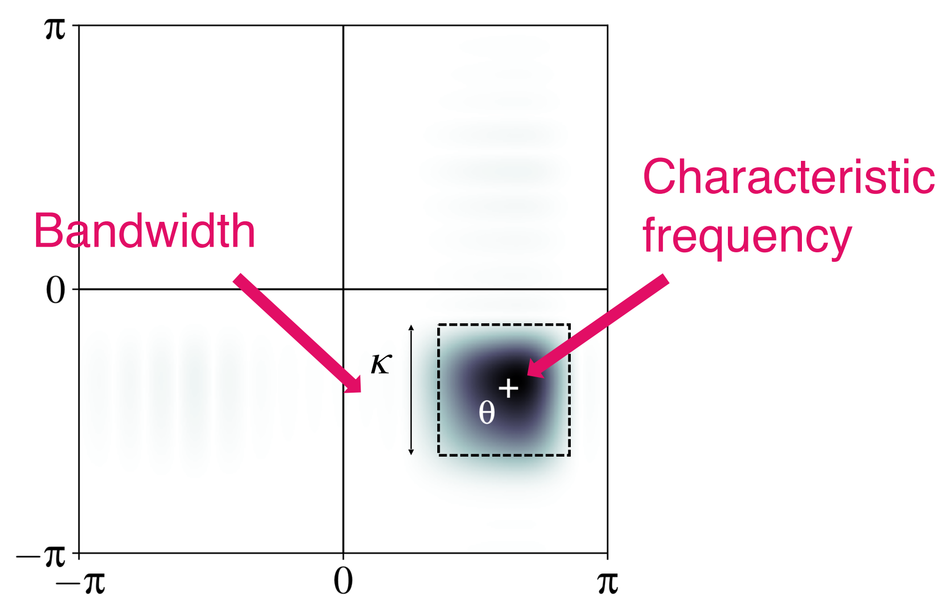
$\mathrm{V} =$





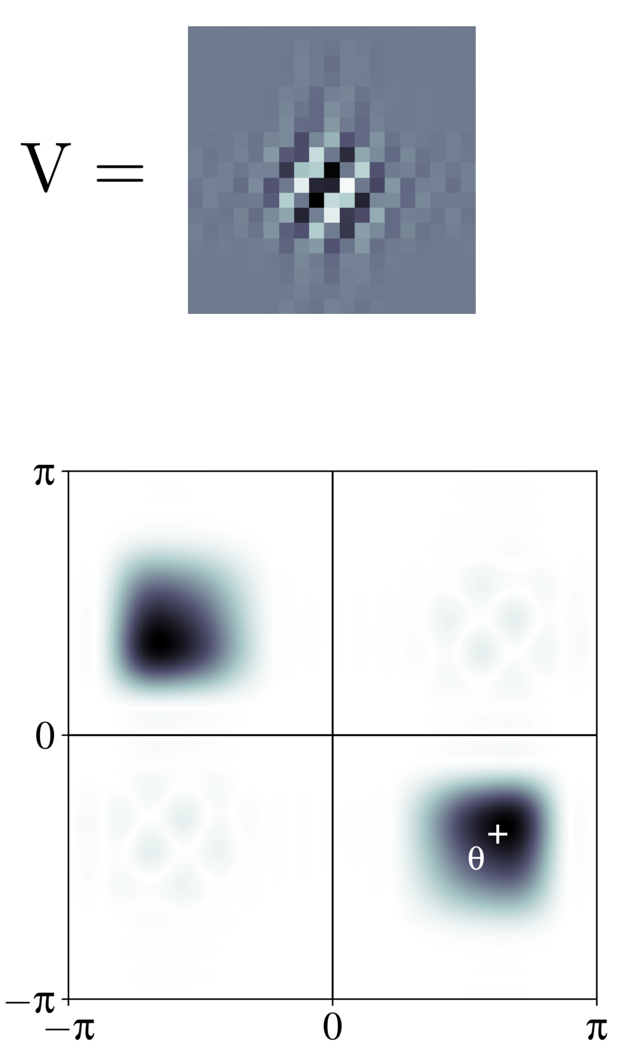
$\mathrm{W} = \mathrm{V} + i\,\mathcal{H}(\mathrm{V})$

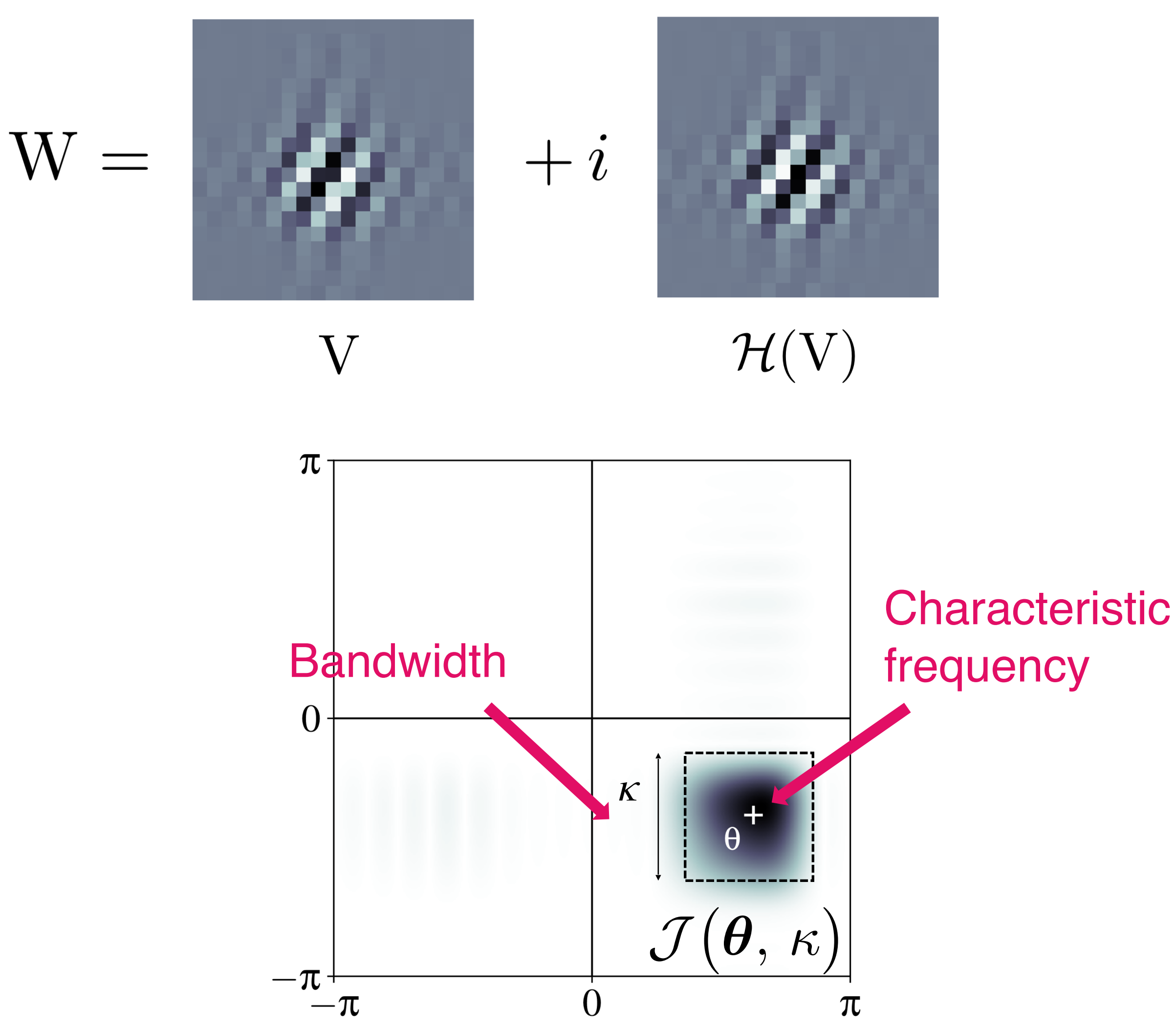




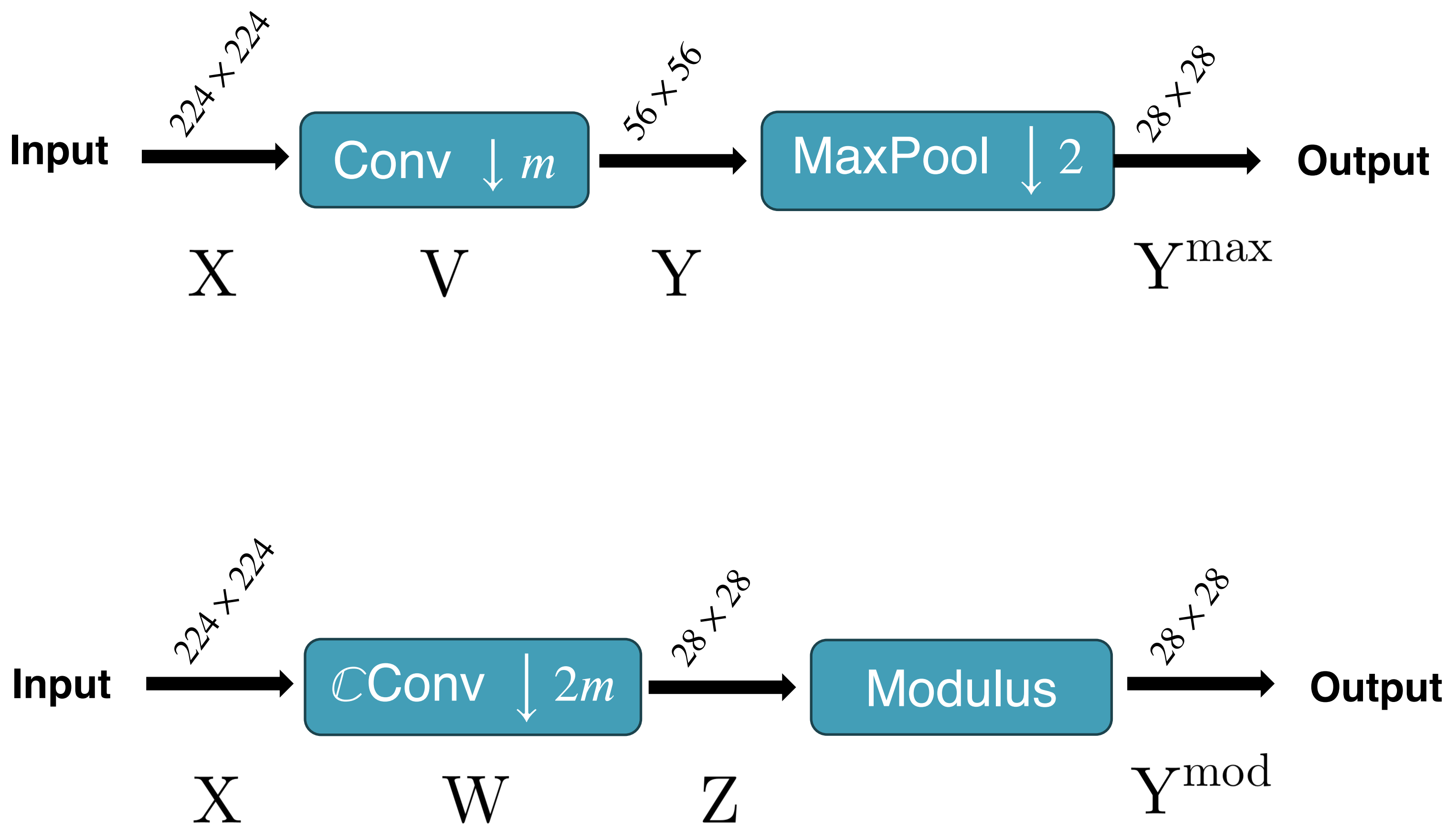
Fundamental hypothesis on filter

- Band-pass, oriented and **analytic** *Gabor-like* filters W

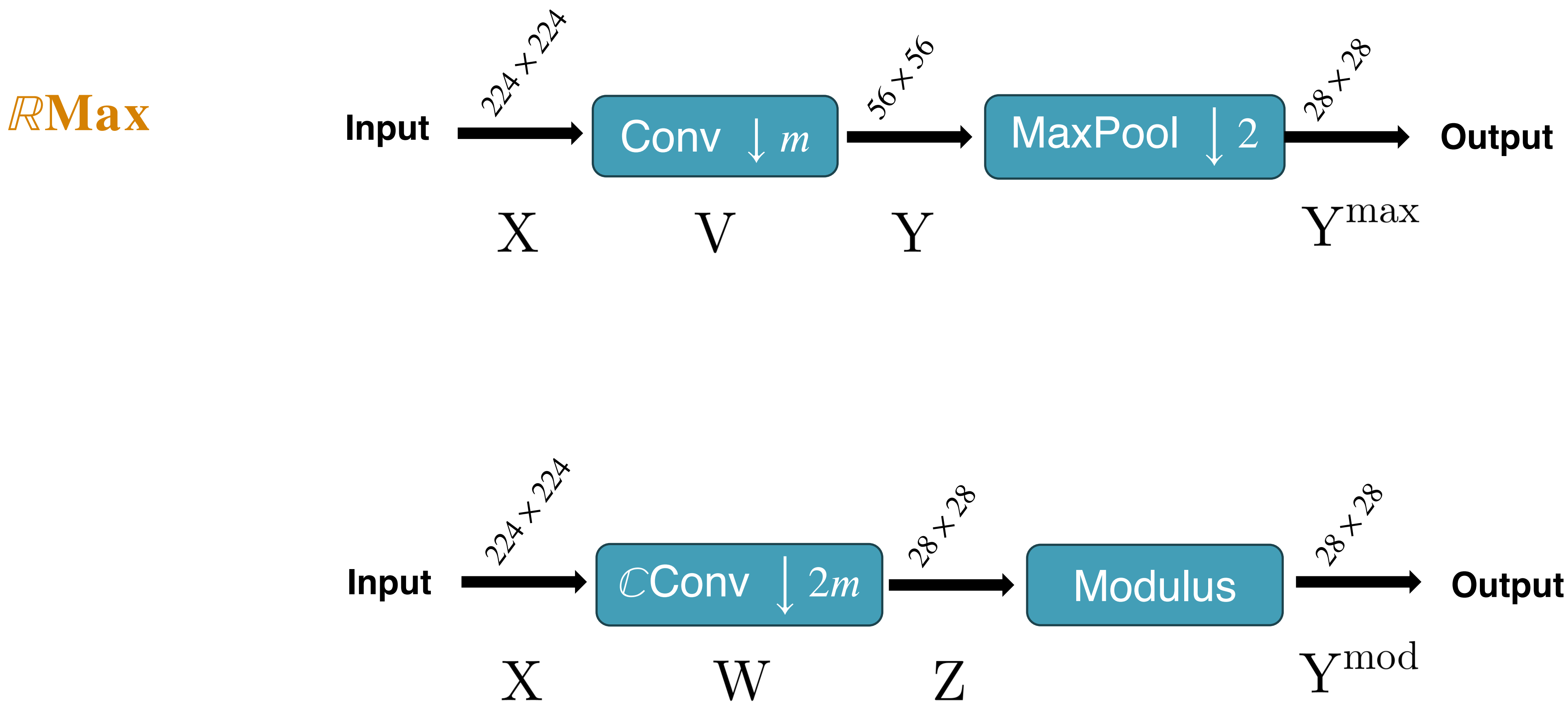




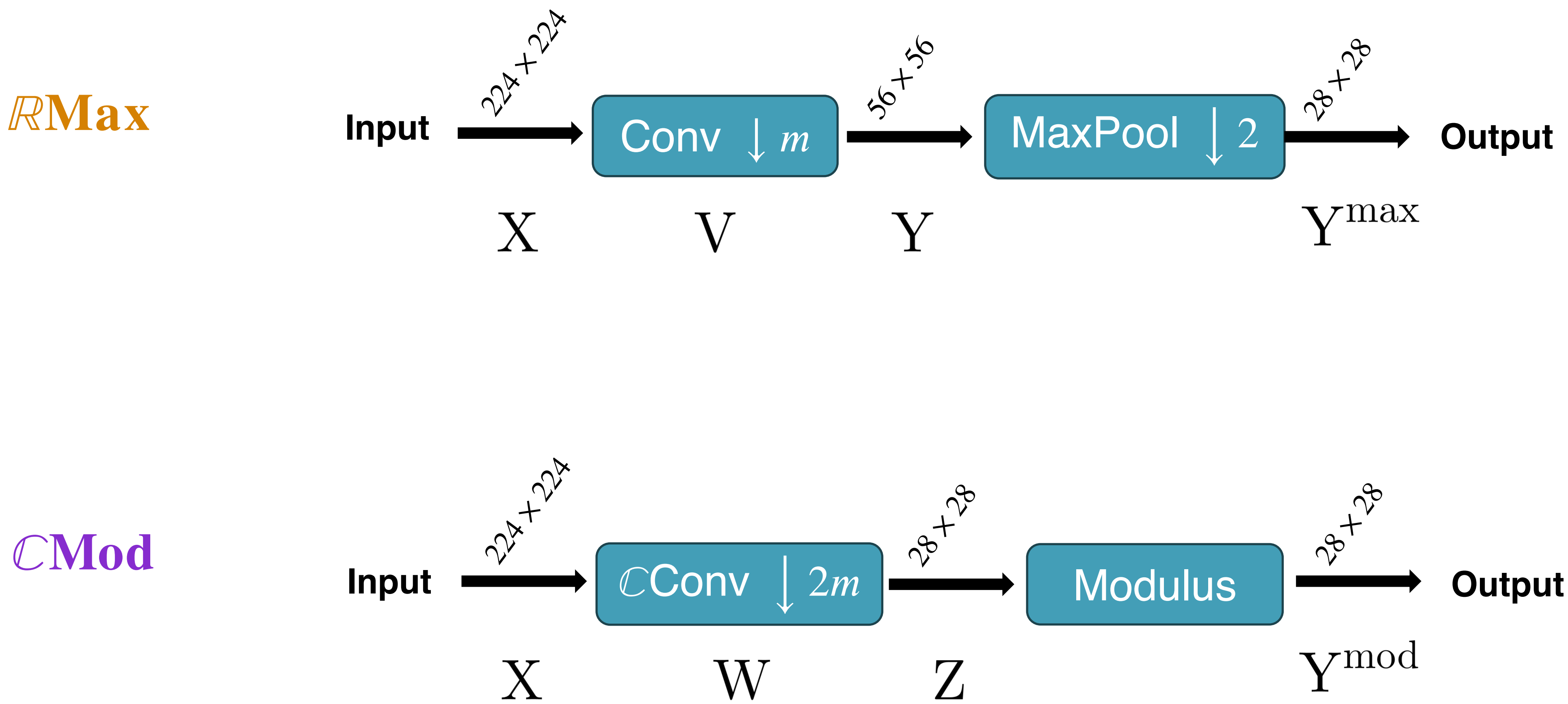
Two operators to compare



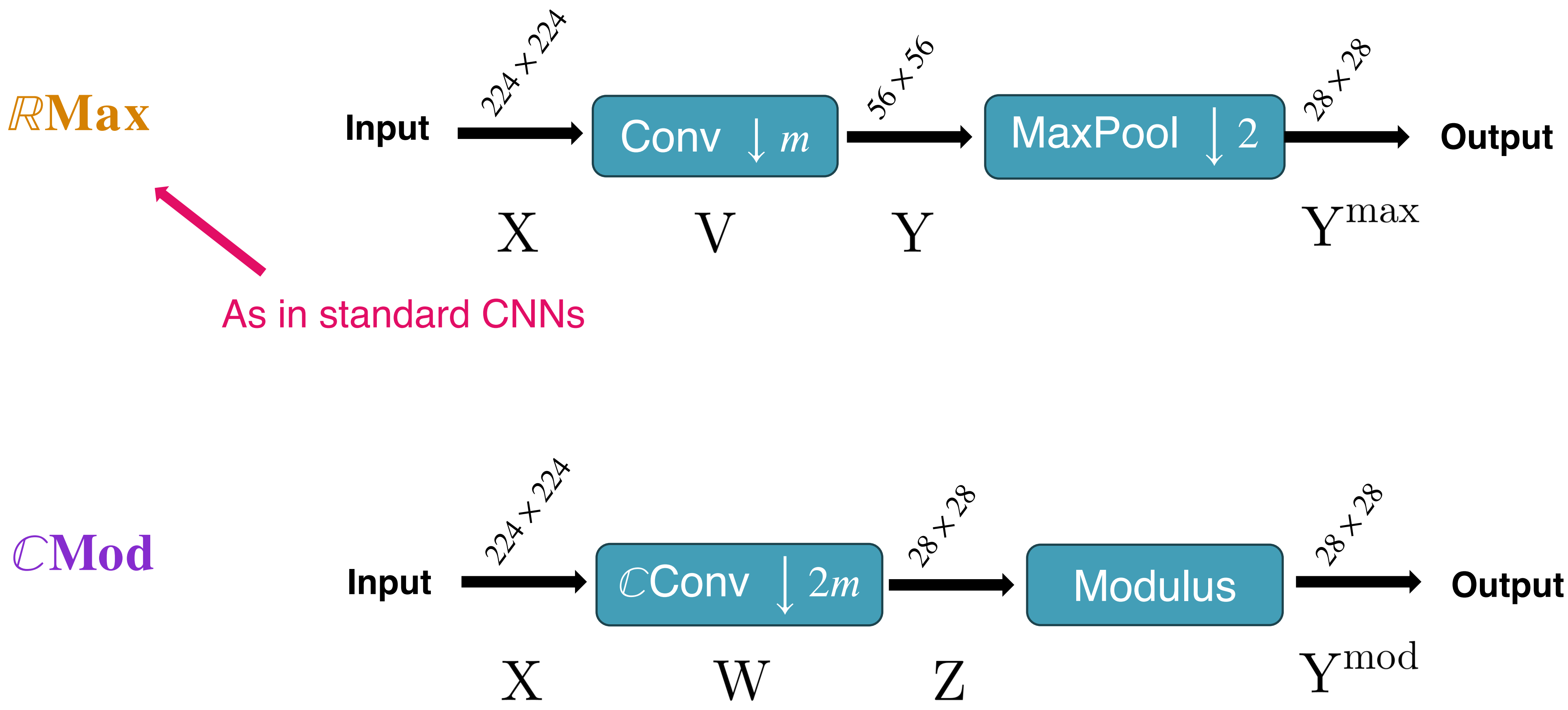
Two operators to compare



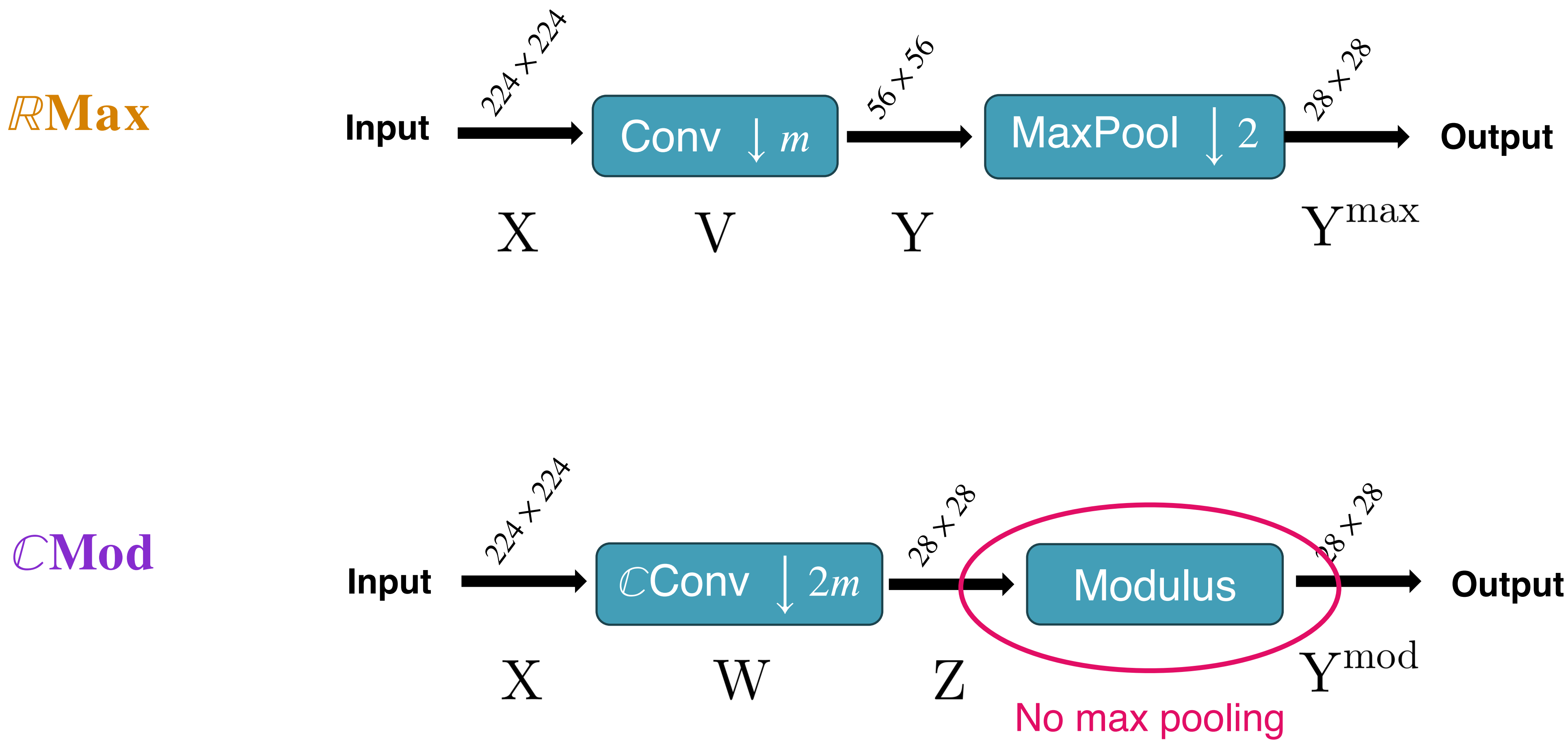
Two operators to compare



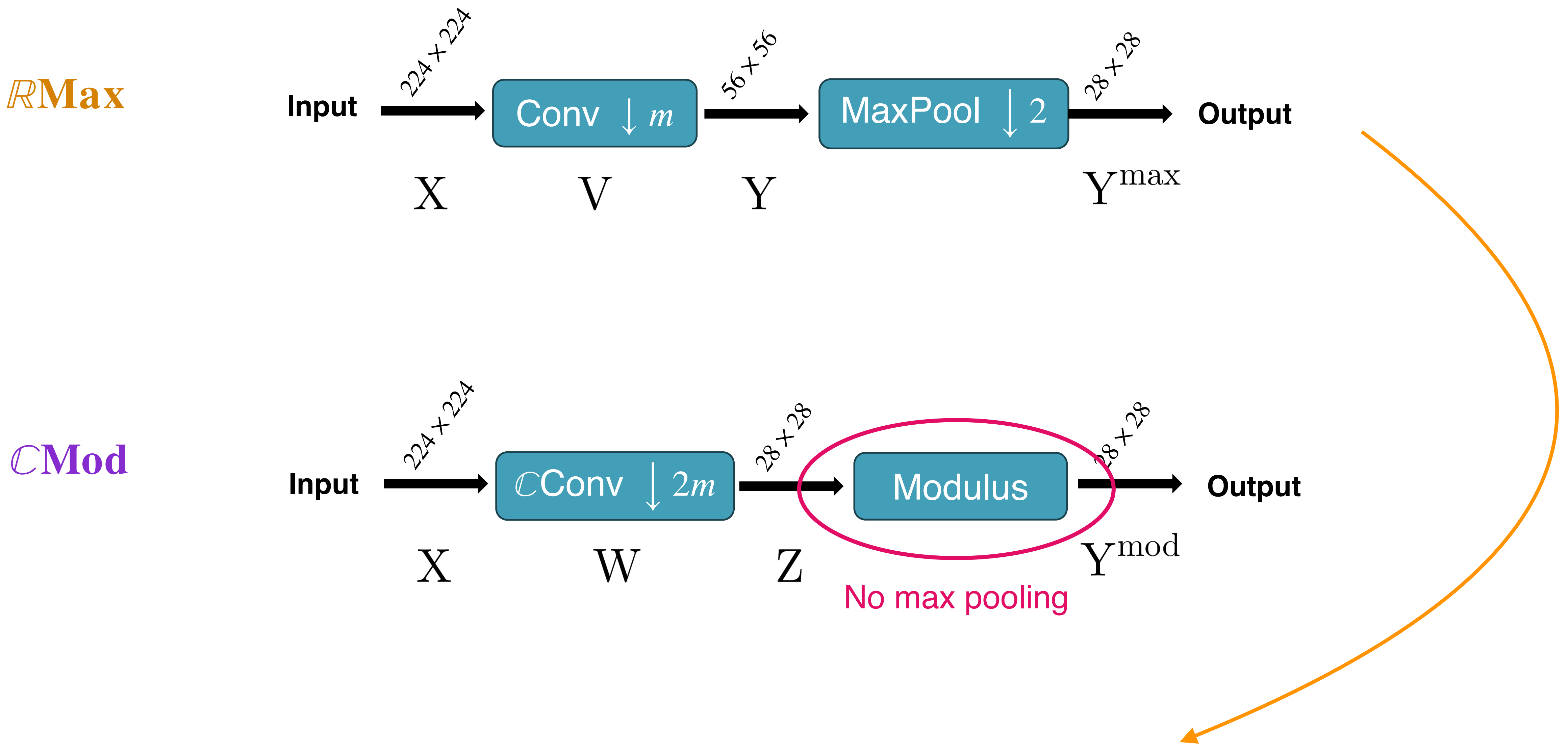
Two operators to compare



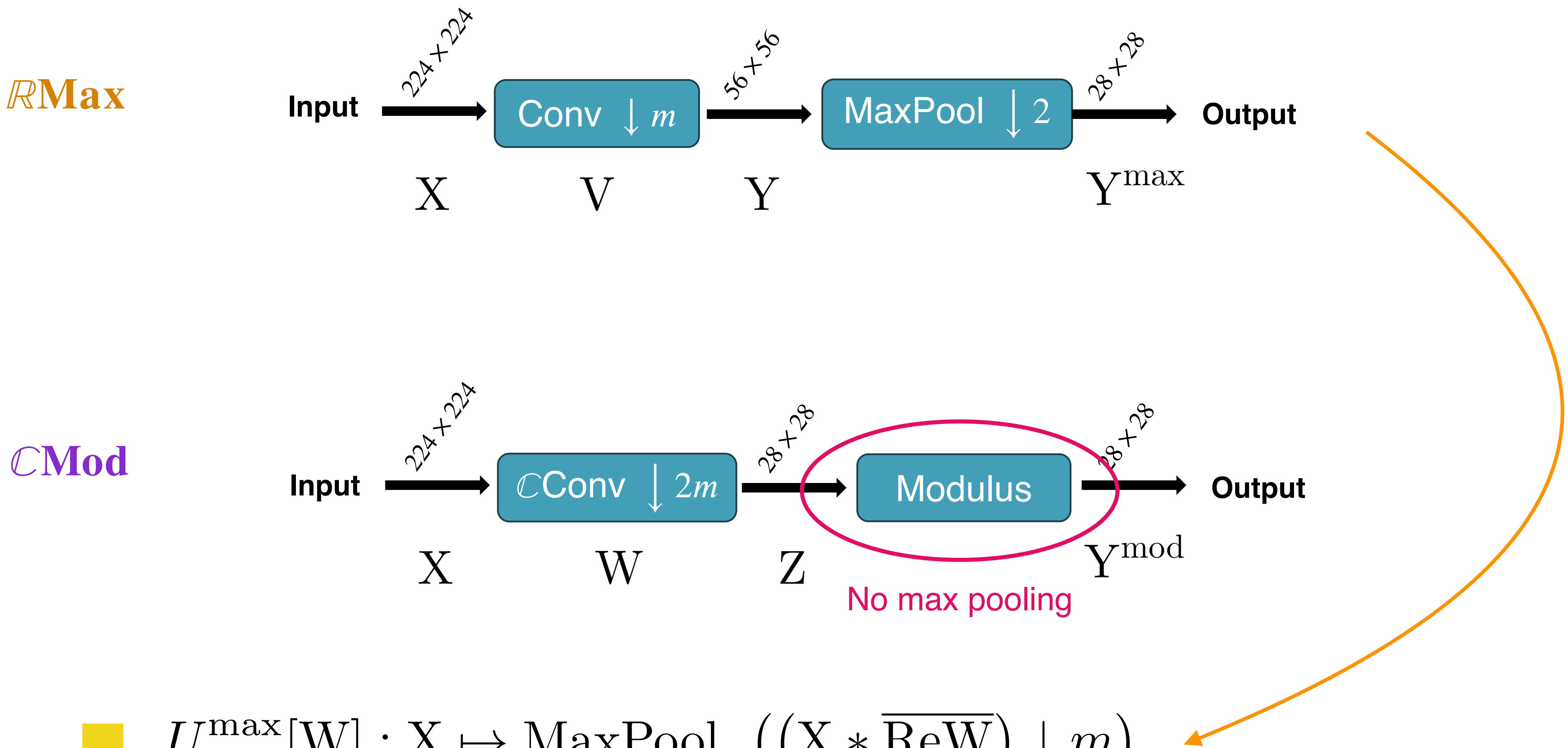
Two operators to compare



Two operators to compare

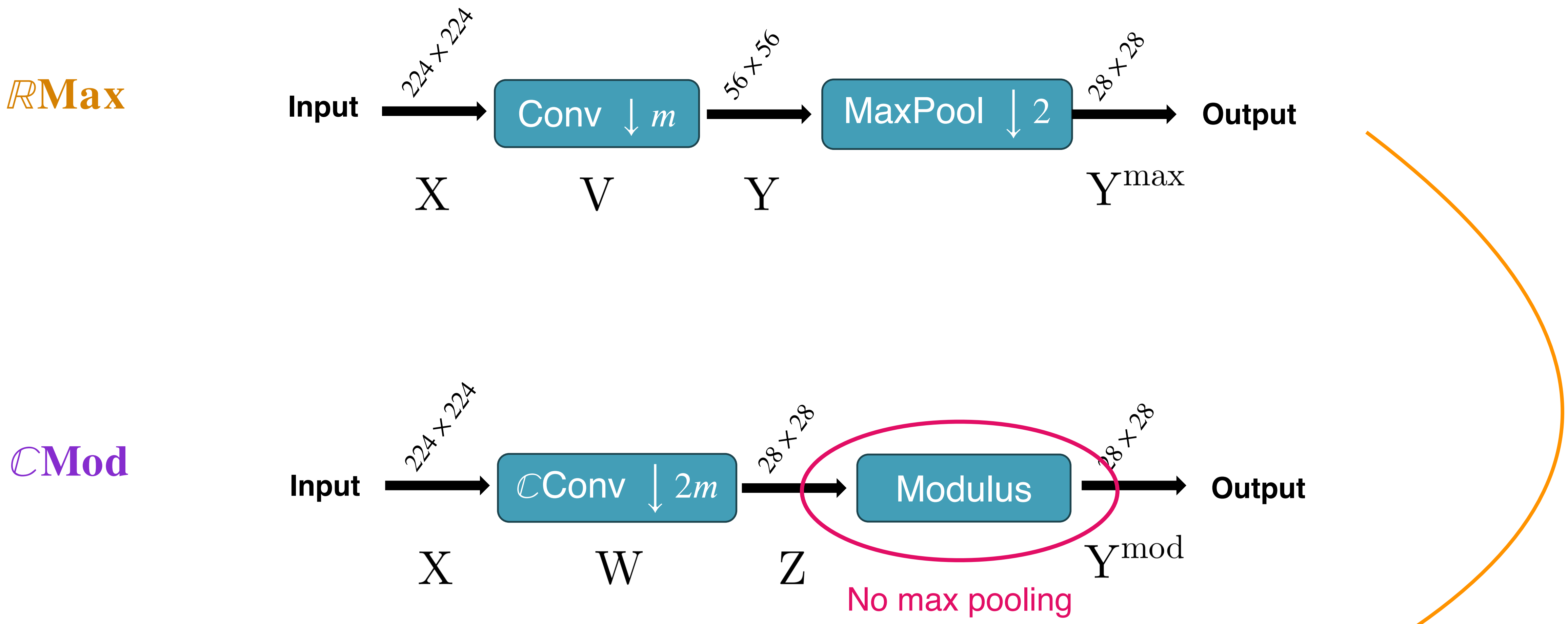


Two operators to compare



- $U_{m,q}^{\max}[\mathrm{W}] : \mathrm{X} \mapsto \mathrm{MaxPool}_q\left(\left(\mathrm{X} * \overline{\mathrm{Re}\mathrm{W}}\right) \downarrow m\right)$

Two operators to compare

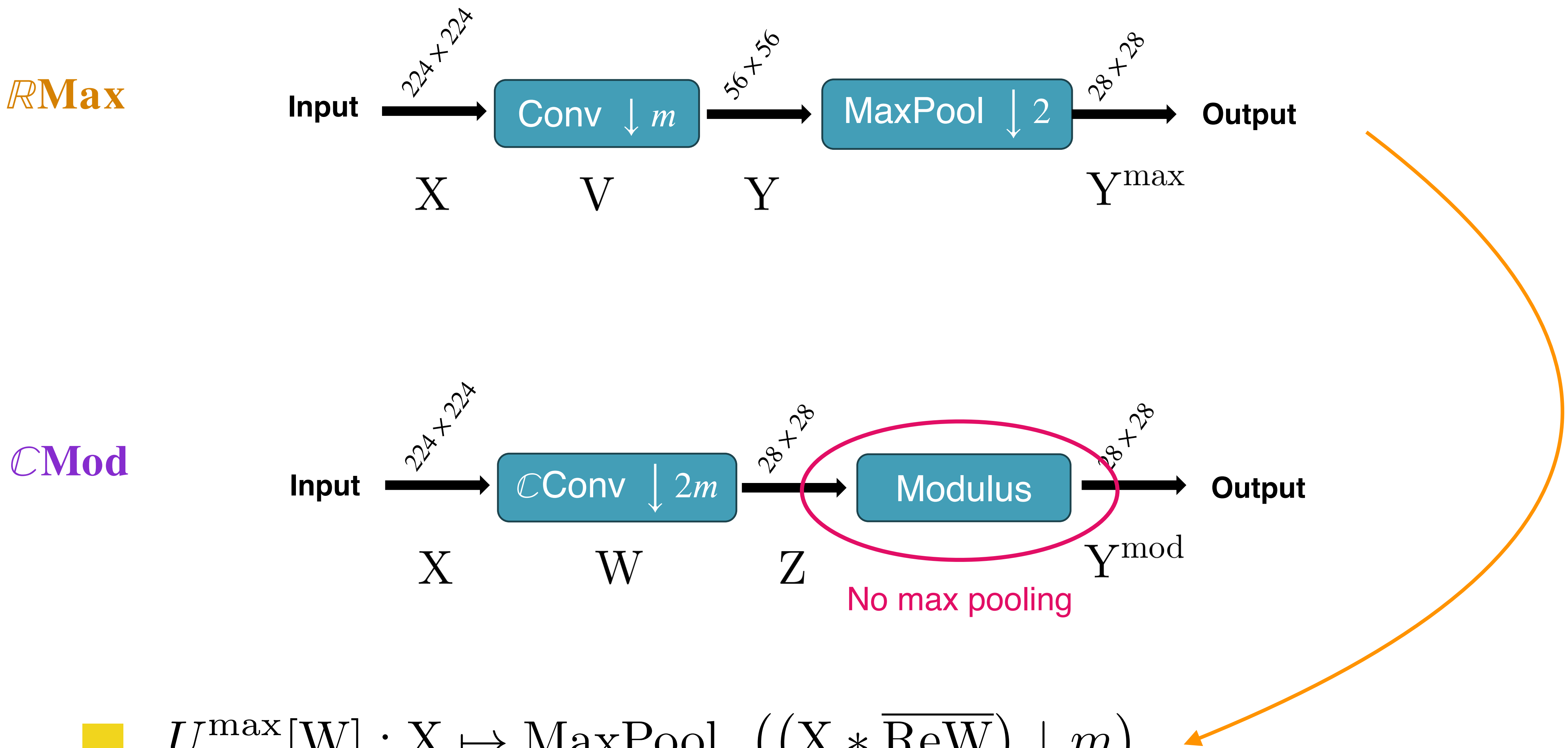


- $U_{m,q}^{\max}[\mathrm{W}] : \mathrm{X} \mapsto \mathrm{MaxPool}_q\left(\left(\mathrm{X} * \overline{\mathrm{Re}\mathrm{W}}\right) \downarrow m\right)$

$$(\mathrm{X} * \overline{\mathrm{V}})[\boldsymbol{n}] := \sum_{\boldsymbol{p} \in \mathbb{Z}^2} \mathrm{X}[\boldsymbol{p}]\overline{\mathrm{V}}[\boldsymbol{n} - \boldsymbol{p}] \qquad \overline{\mathrm{V}}[\boldsymbol{n}] := \mathrm{V}[-\boldsymbol{n}]$$

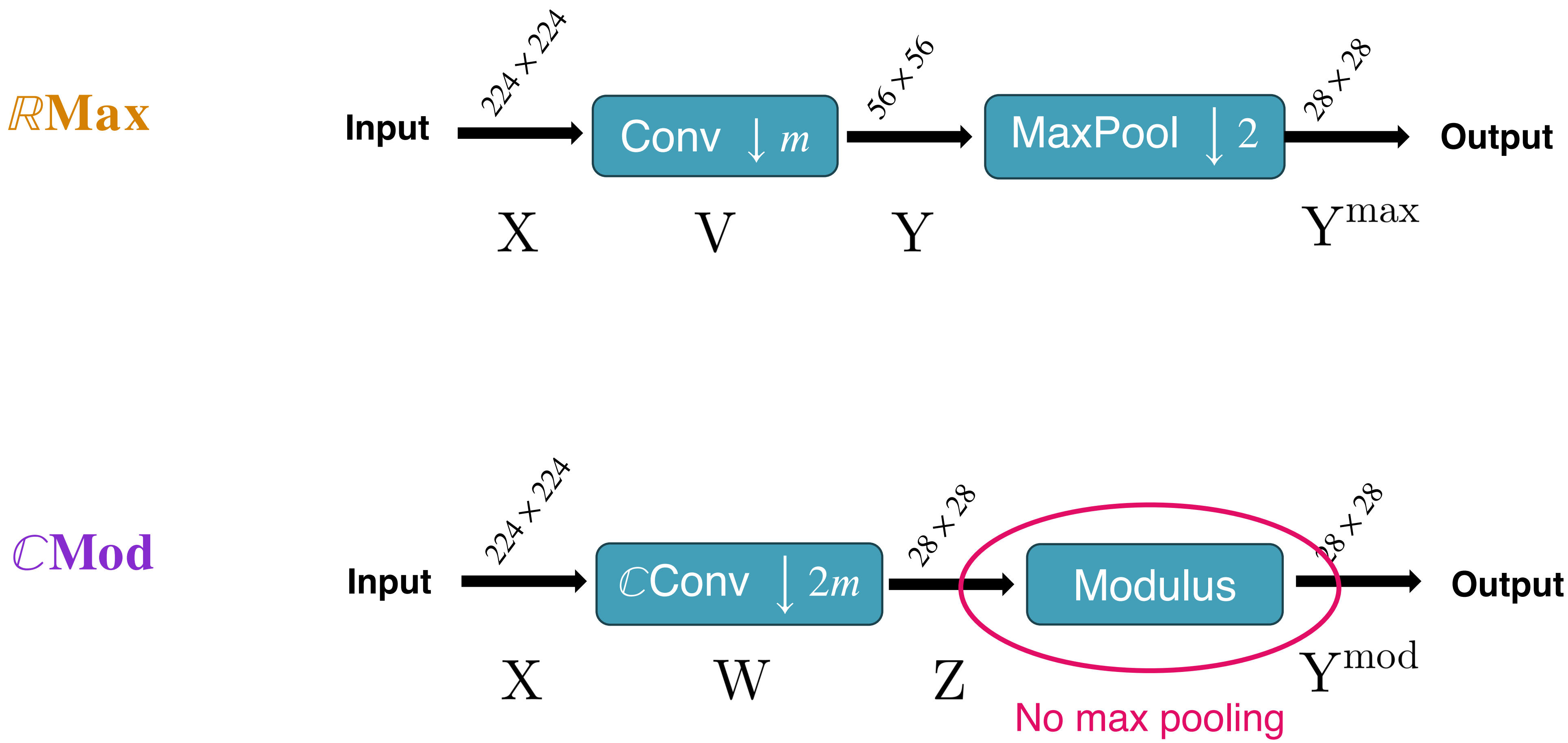
$$\mathrm{MaxPool}_q(\mathrm{Y})[\boldsymbol{n}] := \max_{\|\boldsymbol{p}\|_\infty \leq q} \mathrm{Y}[2\boldsymbol{n} + \boldsymbol{p}] \qquad (\mathrm{Y} \downarrow m)[\boldsymbol{n}] := \mathrm{Y}[m\boldsymbol{n}]$$

Two operators to compare



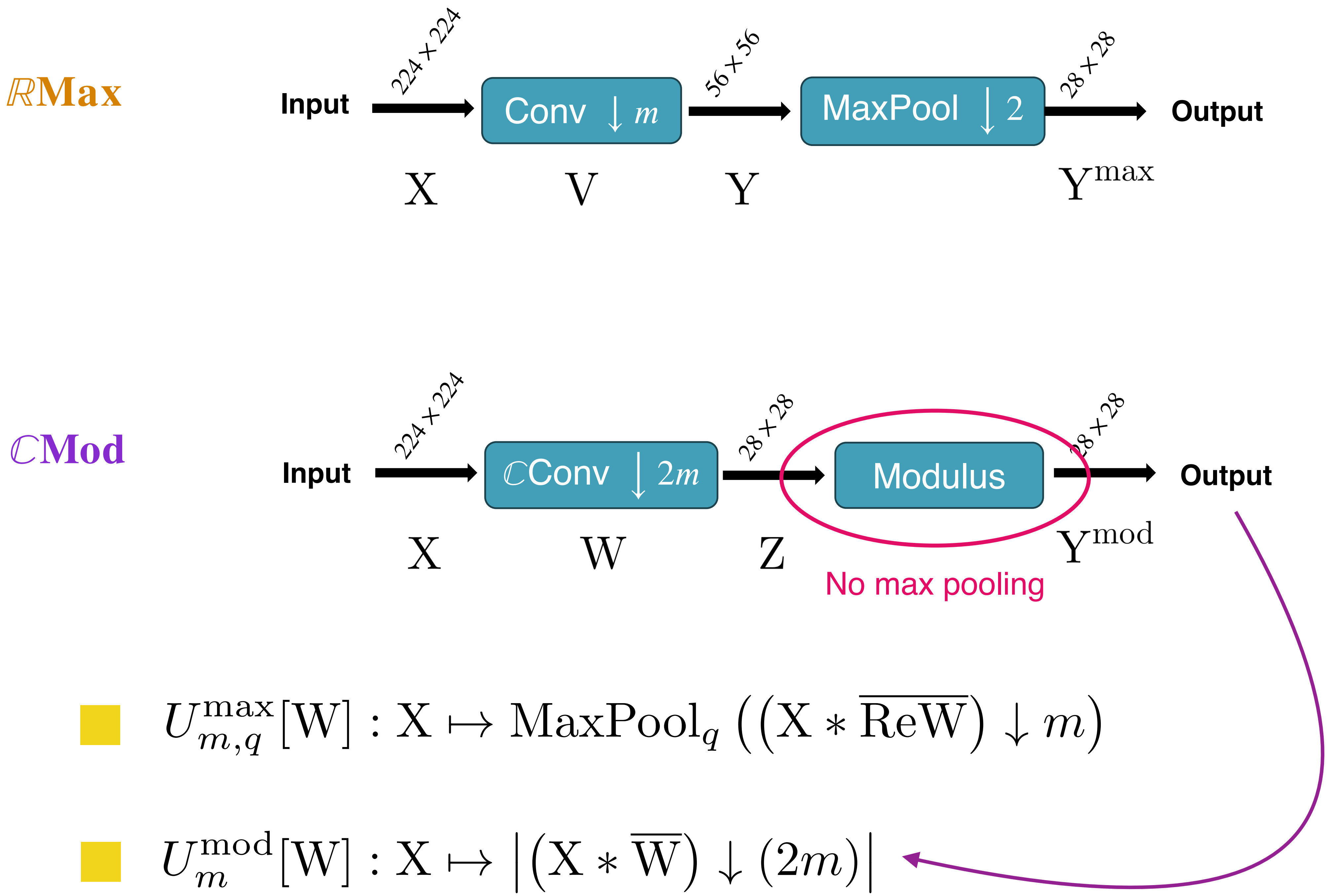
- $U_{m,q}^{\max}[\mathrm{W}] : \mathrm{X} \mapsto \mathrm{MaxPool}_q \left(\left(\mathrm{X} * \overline{\mathrm{Re W}} \right) \downarrow m \right)$

Two operators to compare



- $U^{\max}_{m,q}[\mathrm{W}] : \mathrm{X} \mapsto \mathrm{MaxPool}_q\left(\left(\mathrm{X} * \overline{\mathrm{Re W}}\right) \downarrow m\right)$

Two operators to compare



- $U^{\max}_{m,q}[\mathrm{W}] : \mathrm{X} \mapsto \mathrm{MaxPool}_q\left(\left(\mathrm{X} * \overline{\mathrm{Re}\mathrm{W}}\right) \downarrow m\right)$

- $U^{\mathrm{mod}}_{m}[\mathrm{W}] : \mathrm{X} \mapsto \left|\left(\mathrm{X} * \overline{\mathrm{W}}\right) \downarrow (2m)\right|$

Roadmap

Roadmap

- Show that, **under the Gabor hypothesis**, $\mathbb{C}\mathbf{Mod}$ is **stable** with respect to small input shifts

Roadmap

- Show that, **under the Gabor hypothesis**, $\mathbb{C}$**Mod** is **stable** with respect to small input shifts

- **Establish conditions** on the filter's frequency and orientation under which $\mathbb{R}$**Max** and $\mathbb{C}$**Mod** **produce comparable outputs**:

$$U_{m,q}^{\max}[\mathrm{W}](\mathrm{X}) \approx U_{m}^{\mathrm{mod}}[\mathrm{W}](\mathrm{X})$$

Roadmap

- Show that, **under the Gabor hypothesis**, $\mathbb{C}\mathbf{Mod}$ is **stable** with respect to small input shifts

- **Establish conditions** on the filter's frequency and orientation under which $\mathbb{R}\mathbf{Max}$ and $\mathbb{C}\mathbf{Mod}$ **produce comparable outputs**:

$$U_{m,q}^{\max}[\mathrm{W}](\mathrm{X}) \approx U_{m}^{\mathrm{mod}}[\mathrm{W}](\mathrm{X})$$

- Deduce a **measure of shift invariance** for $\mathbb{R}\mathbf{Max}$ operator, which benefits from the stability of $\mathbb{C}\mathbf{Mod}$

Roadmap

- Show that, **under the Gabor hypothesis**, $\mathbb{C}\mathbf{Mod}$ is **stable** with respect to small input shifts

- **Establish conditions** on the filter's frequency and orientation under which $\mathbb{R}\mathbf{Max}$ and $\mathbb{C}\mathbf{Mod}$ **produce comparable outputs**:

$$U_{m,q}^{\max}[\mathrm{W}](\mathrm{X}) \approx U_{m}^{\mathrm{mod}}[\mathrm{W}](\mathrm{X})$$

- Deduce a **measure of shift invariance** for $\mathbb{R}\mathbf{Max}$ operator, which benefits from the stability of $\mathbb{C}\mathbf{Mod}$

- Extend our results to **multichannel operators** (RGB images), such as implemented in conventional CNN architectures

Roadmap

- Show that, **under the Gabor hypothesis**, $\mathbb{C}\mathbf{Mod}$ is **stable** with respect to small input shifts

- **Establish conditions** on the filter's frequency and orientation under which $\mathbb{R}\mathbf{Max}$ and $\mathbb{C}\mathbf{Mod}$ **produce comparable outputs**:

$$U_{m,q}^{\max}[\mathrm{W}](\mathrm{X}) \approx U_{m}^{\mathrm{mod}}[\mathrm{W}](\mathrm{X})$$

- Deduce a **measure of shift invariance** for $\mathbb{R}\mathbf{Max}$ operator, which benefits from the stability of $\mathbb{C}\mathbf{Mod}$

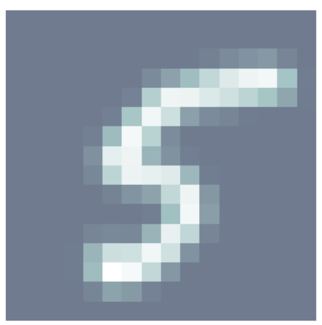
- Extend our results to **multichannel operators** (RGB images), such as implemented in conventional CNN architectures

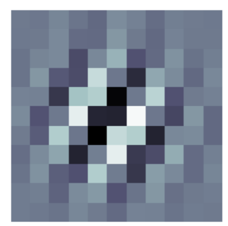
- Experimental validation on a deterministic setting based on the **dual-tree** complex wavelet packet transform (DT-CWPT)

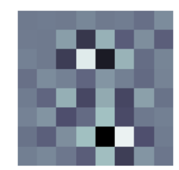
Detour via the continuous framework

- Using the **Shannon-Whittaker sampling theorem**

ℝ**Max**



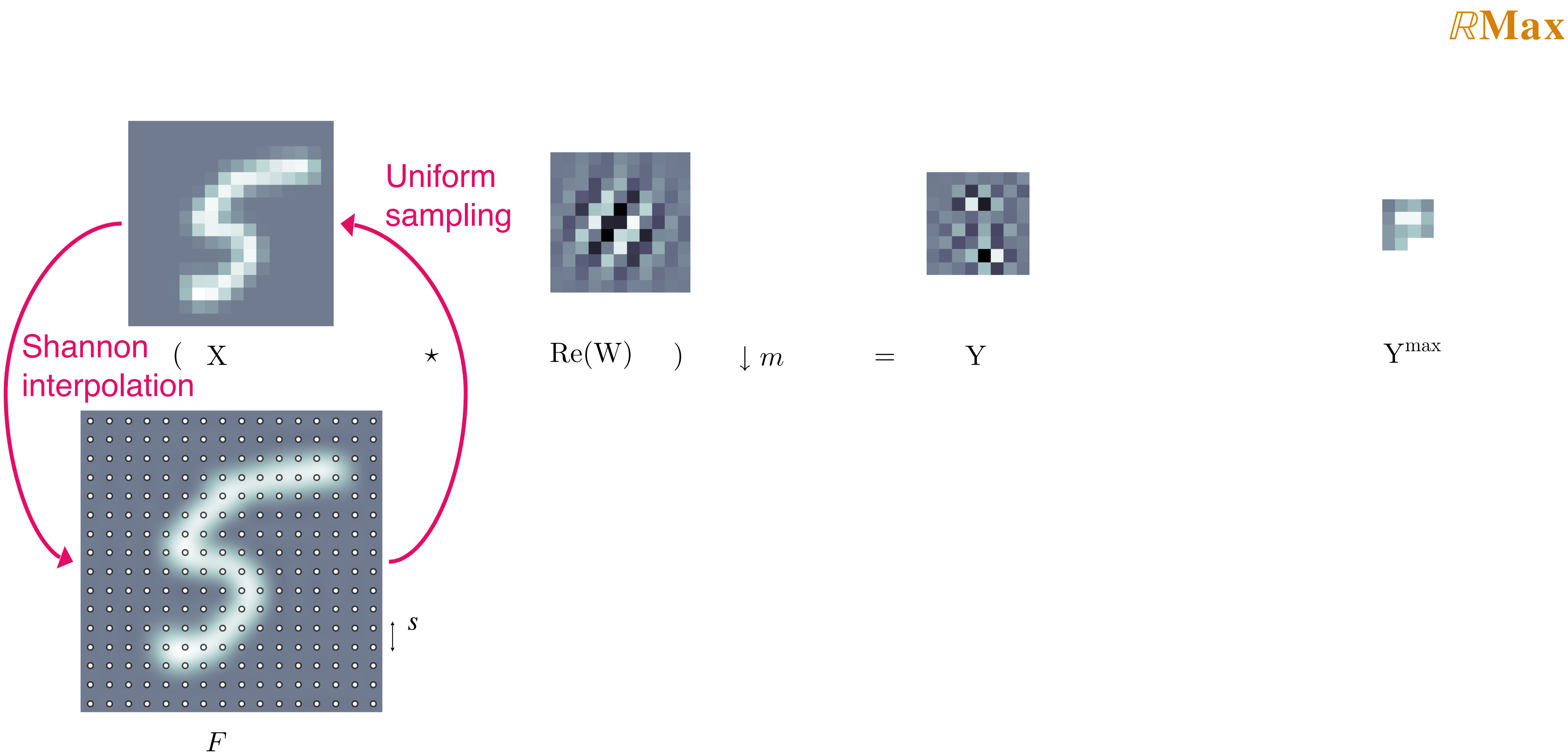




$$(\ \mathrm{X} \star \mathrm{Re(W)}\ ) \downarrow m = \mathrm{Y} \qquad \mathrm{Y}^{\max}$$

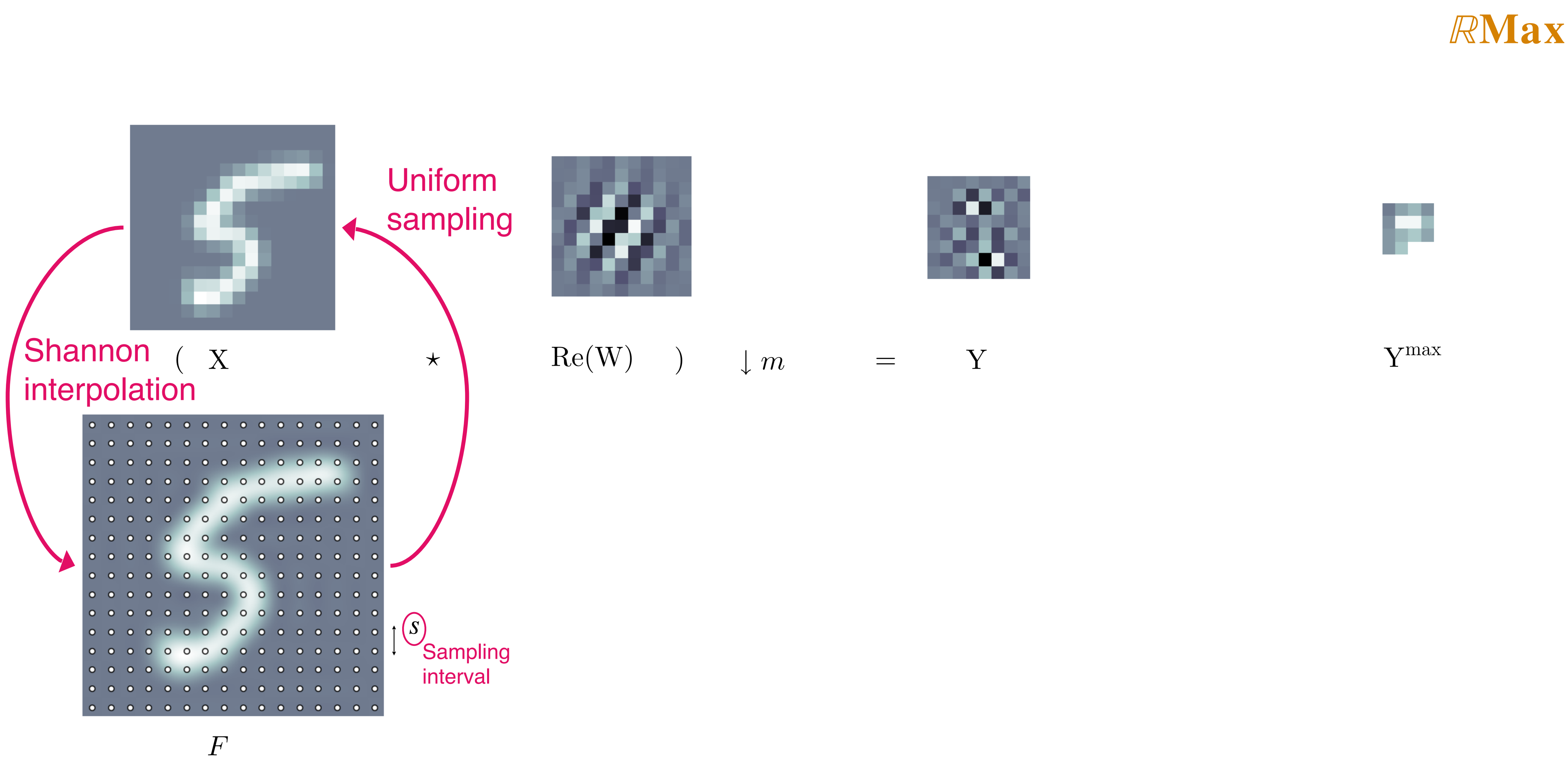
Detour via the continuous framework

- Using the **Shannon-Whittaker sampling theorem**



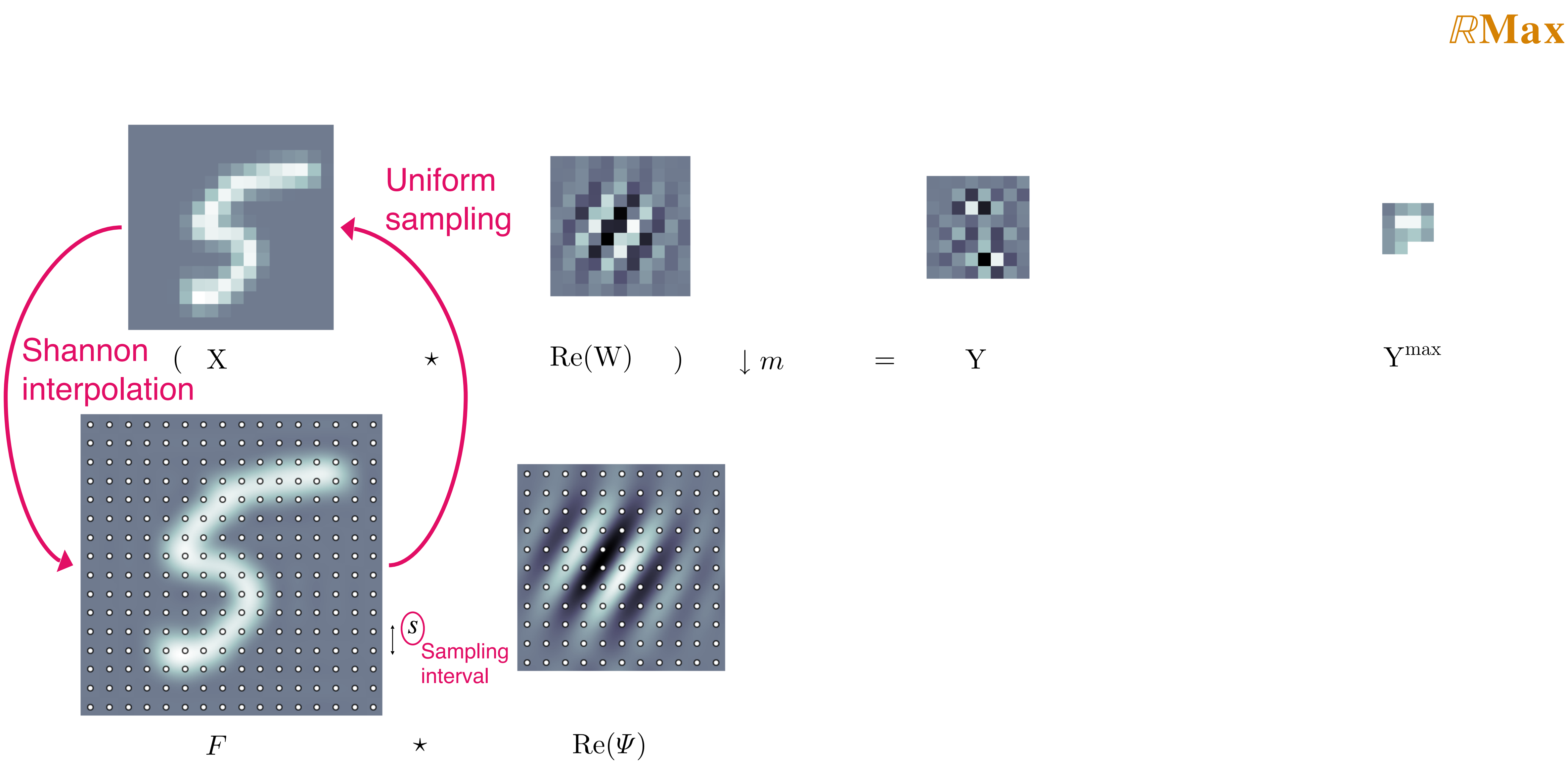
Detour via the continuous framework

- Using the **Shannon-Whittaker sampling theorem**



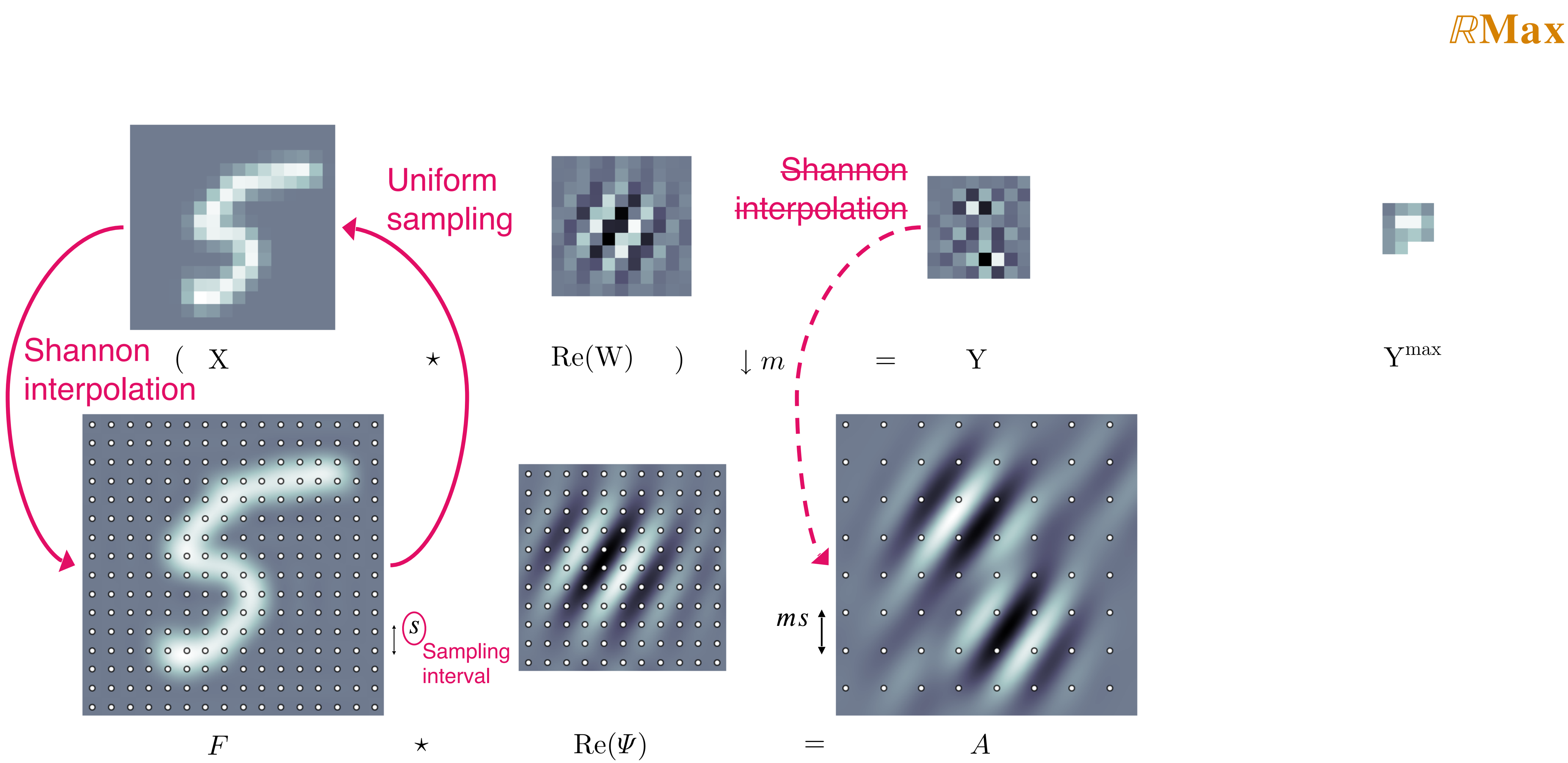
Detour via the continuous framework

- Using the **Shannon-Whittaker sampling theorem**



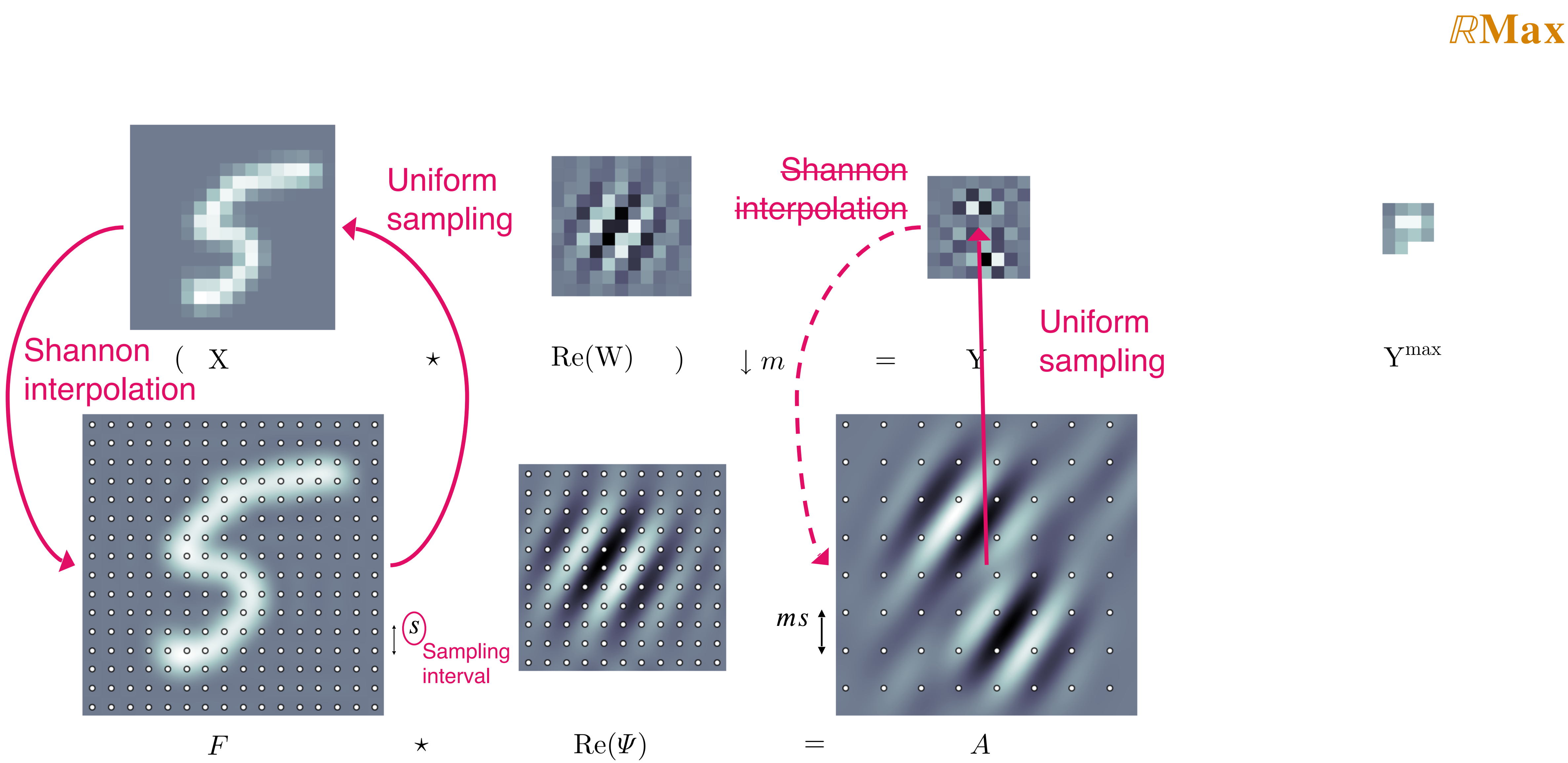
Detour via the continuous framework

- Using the **Shannon-Whittaker sampling theorem**



Detour via the continuous framework

- Using the **Shannon-Whittaker sampling theorem**

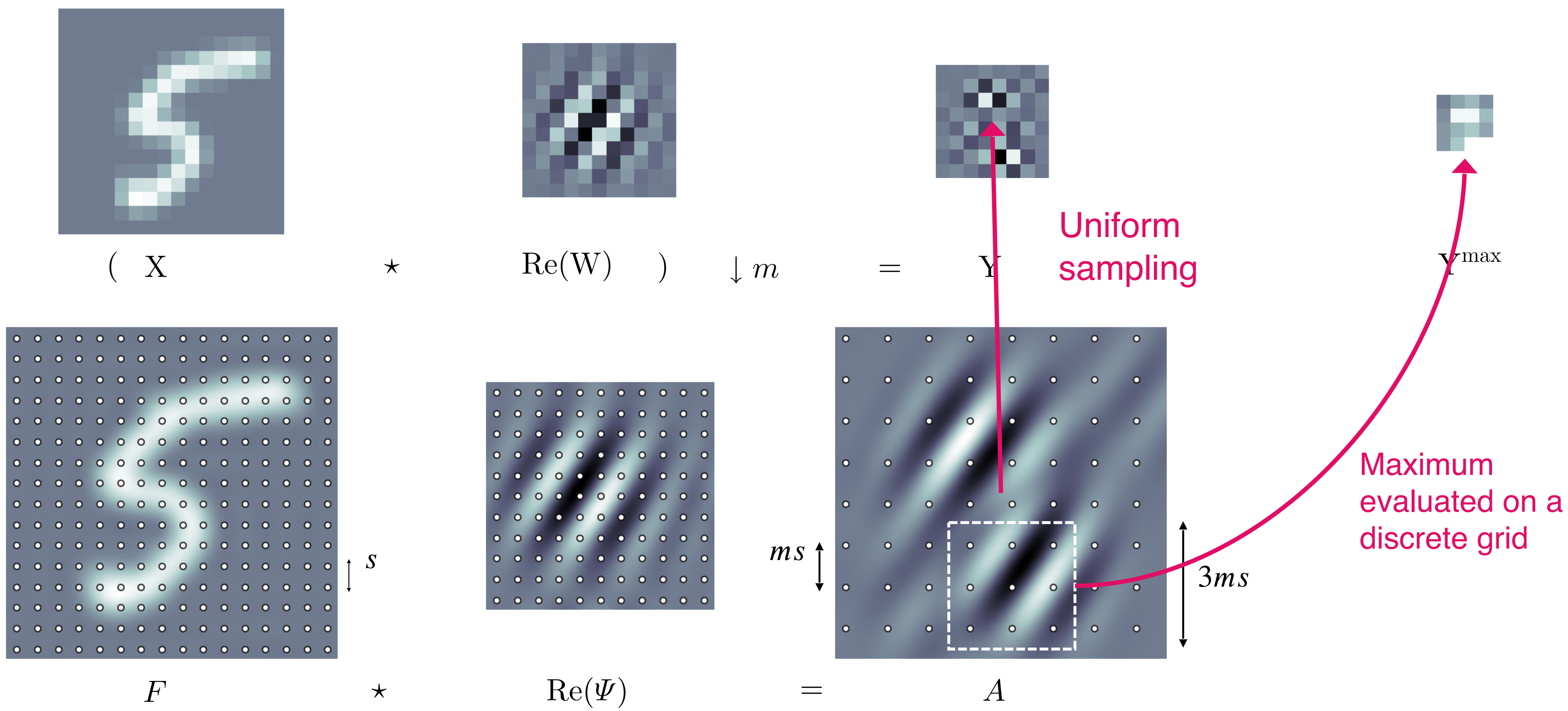


Detour via the continuous framework

- Using the **Shannon-Whittaker sampling theorem**

$$\mathrm{Y}^{\max}[\boldsymbol{n}] = \max_{\|\boldsymbol{p}\|_\infty \leq q} \big(F \star \mathrm{Re}(\Psi)\big)(2ms\boldsymbol{n} + ms\boldsymbol{p})$$

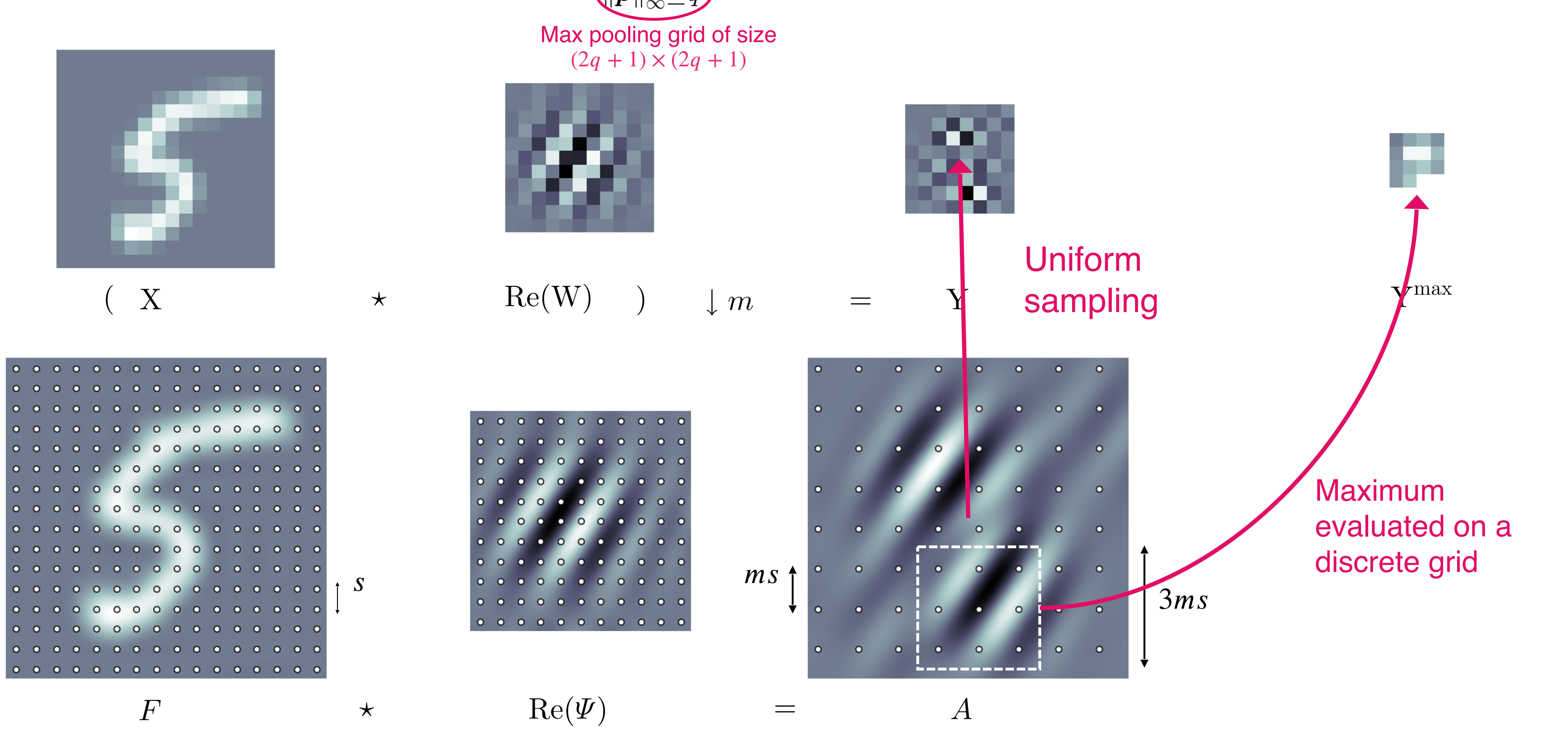
$\mathbb{R}$**Max**



Detour via the continuous framework

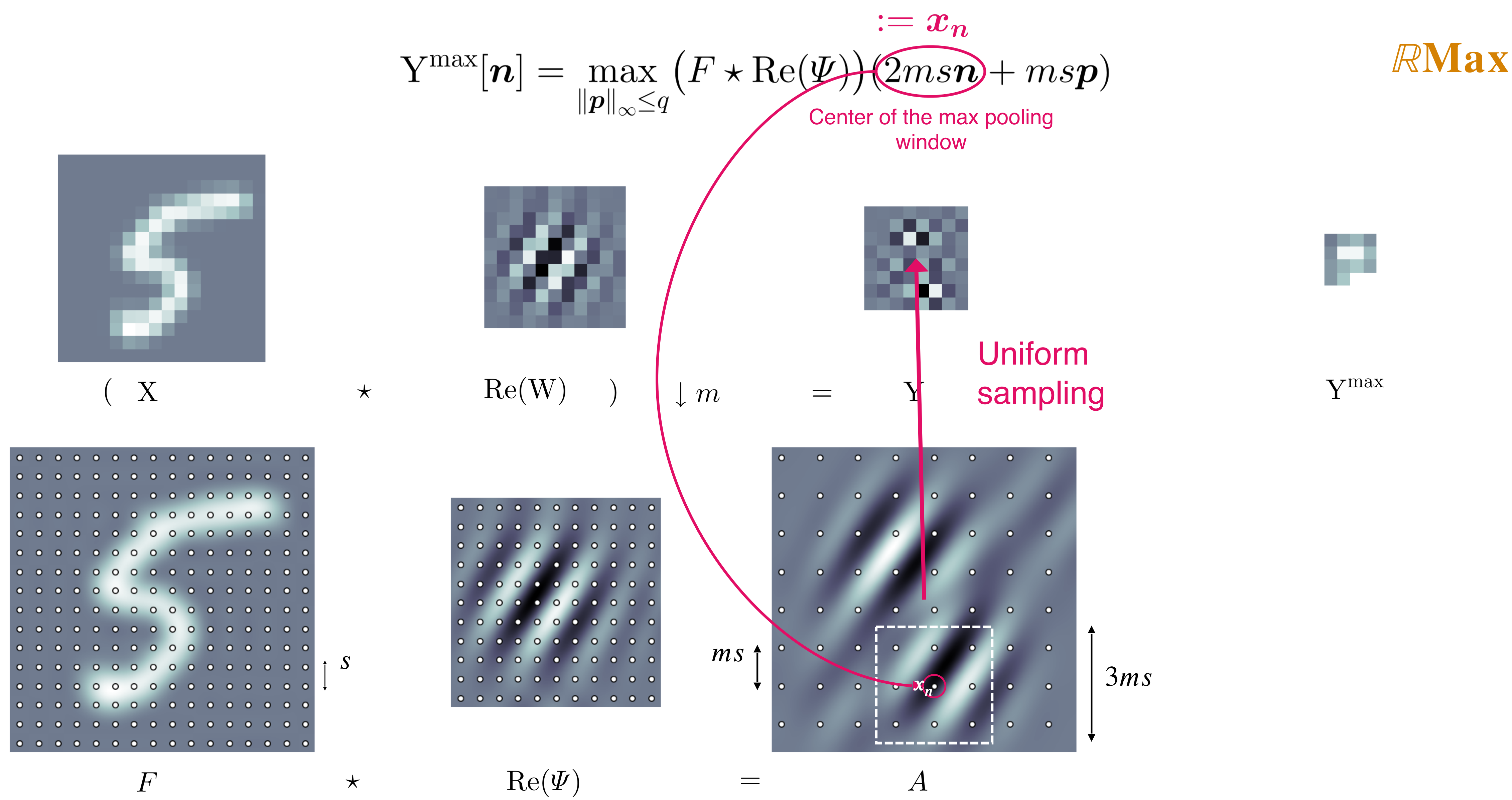
- Using the **Shannon-Whittaker sampling theorem**

$$\mathrm{Y}^{\max}[\boldsymbol{n}] = \max_{\|\boldsymbol{p}\|_\infty \leq q} \big(F \star \mathrm{Re}(\Psi)\big)(2ms\boldsymbol{n} + ms\boldsymbol{p})$$



Detour via the continuous framework

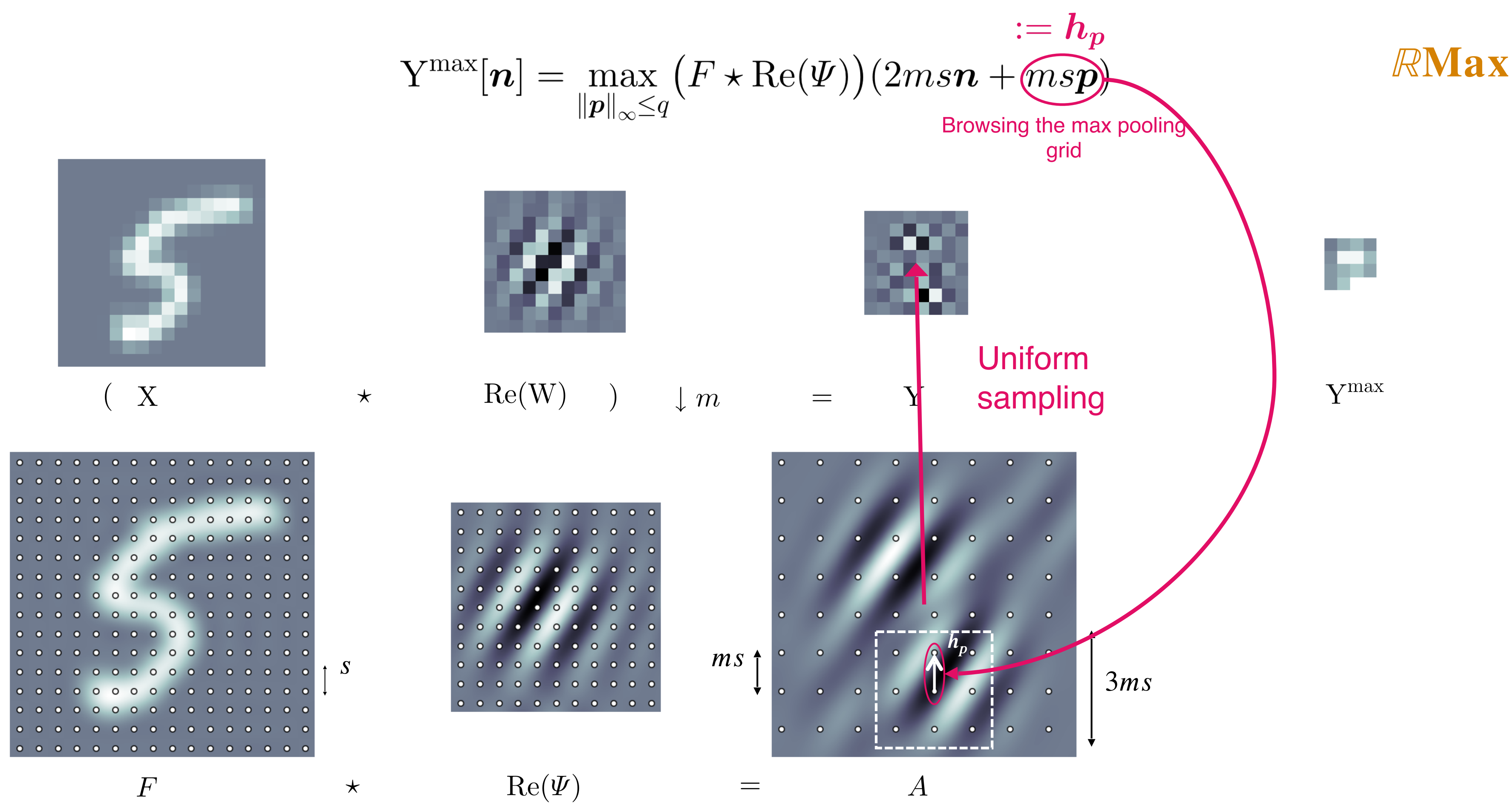
- Using the **Shannon-Whittaker sampling theorem**



Detour via the continuous framework

- Using the **Shannon-Whittaker sampling theorem**

$$\mathrm{Y}^{\max}[\boldsymbol{n}] = \max_{\|\boldsymbol{p}\|_\infty \le q} \big(F \star \mathrm{Re}(\Psi)\big)(2ms\boldsymbol{n} + ms\boldsymbol{p})$$

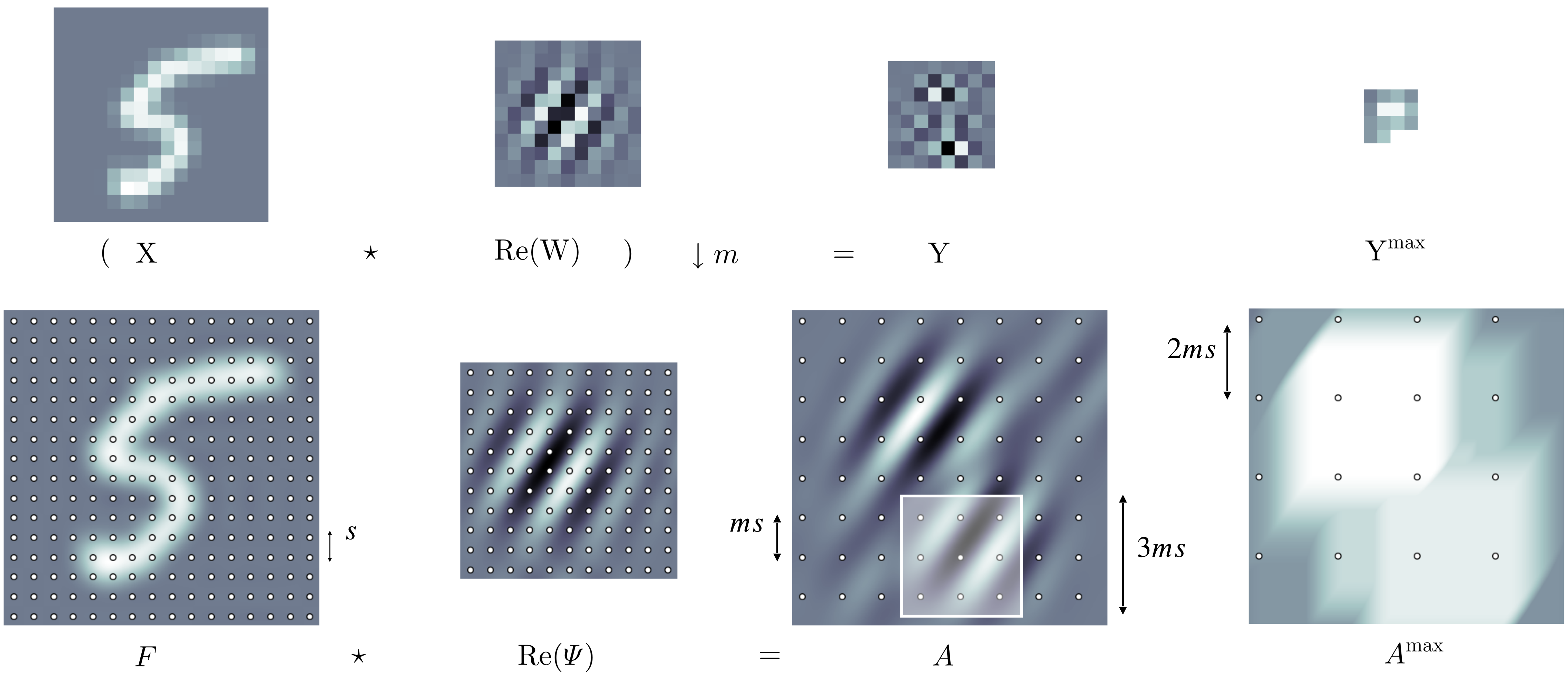


Detour via the continuous framework

- What if we search for the maximum **continuously** in the window?

$$\mathrm{Y}_0^{\max}[\boldsymbol{n}] = \max_{\|\boldsymbol{h}\|_\infty \leq \frac{3ms}{2}} \big(F \star \mathrm{Re}(\Psi)\big)(\boldsymbol{x_n} + \boldsymbol{h}) \approx \mathrm{Y}^{\max}[\boldsymbol{n}]?$$

$\mathbb{R}\mathbf{Max_0}$

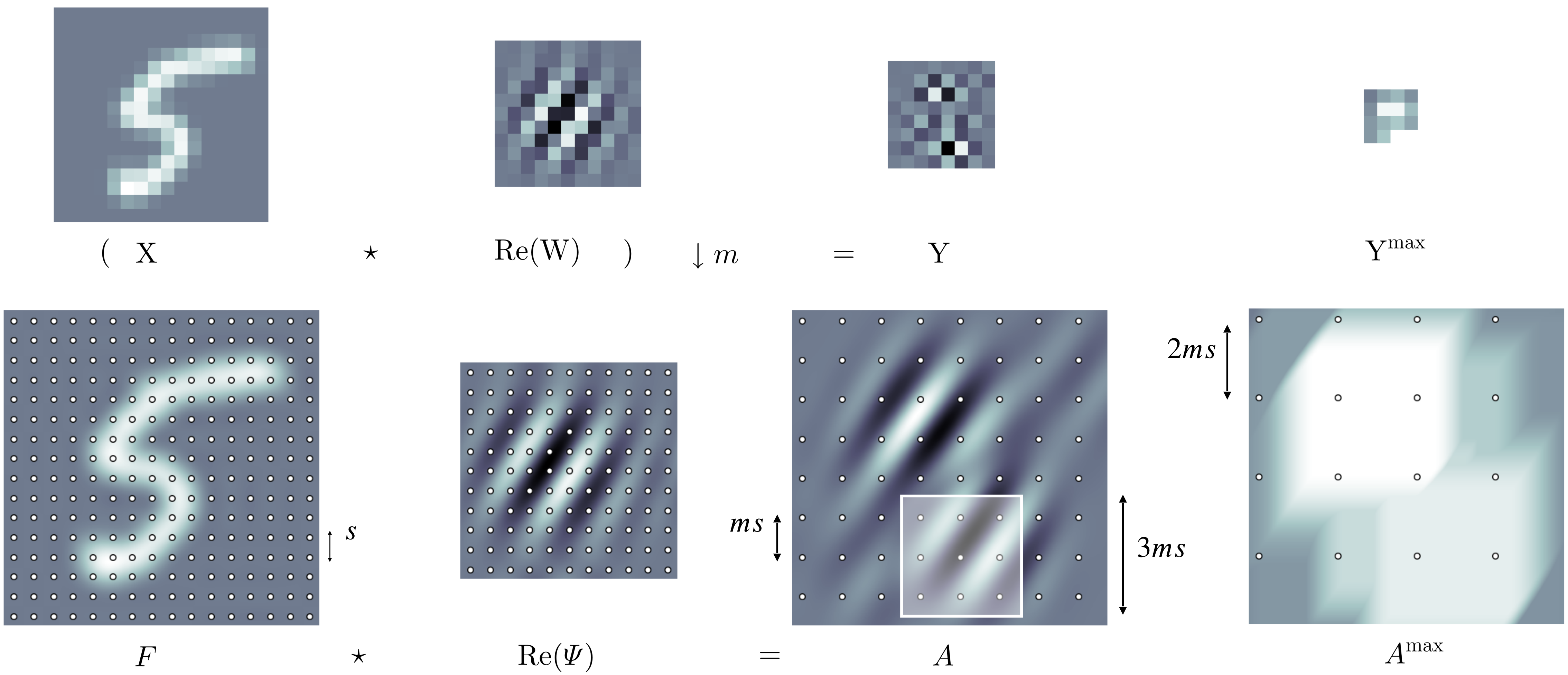


Detour via the continuous framework

- What if we search for the maximum **continuously** in the window?

$$\mathrm{Y}_0^{\max}[\boldsymbol{n}] = \max_{\|\boldsymbol{h}\|_\infty \leq \frac{3ms}{2}} \big(F \star \mathrm{Re}(\Psi)\big)(\boldsymbol{x_n} + \boldsymbol{h}) \approx \mathrm{Y}^{\max}[\boldsymbol{n}]?$$

$\mathbb{R}\mathbf{Max_0}$



Detour via the continuous framework

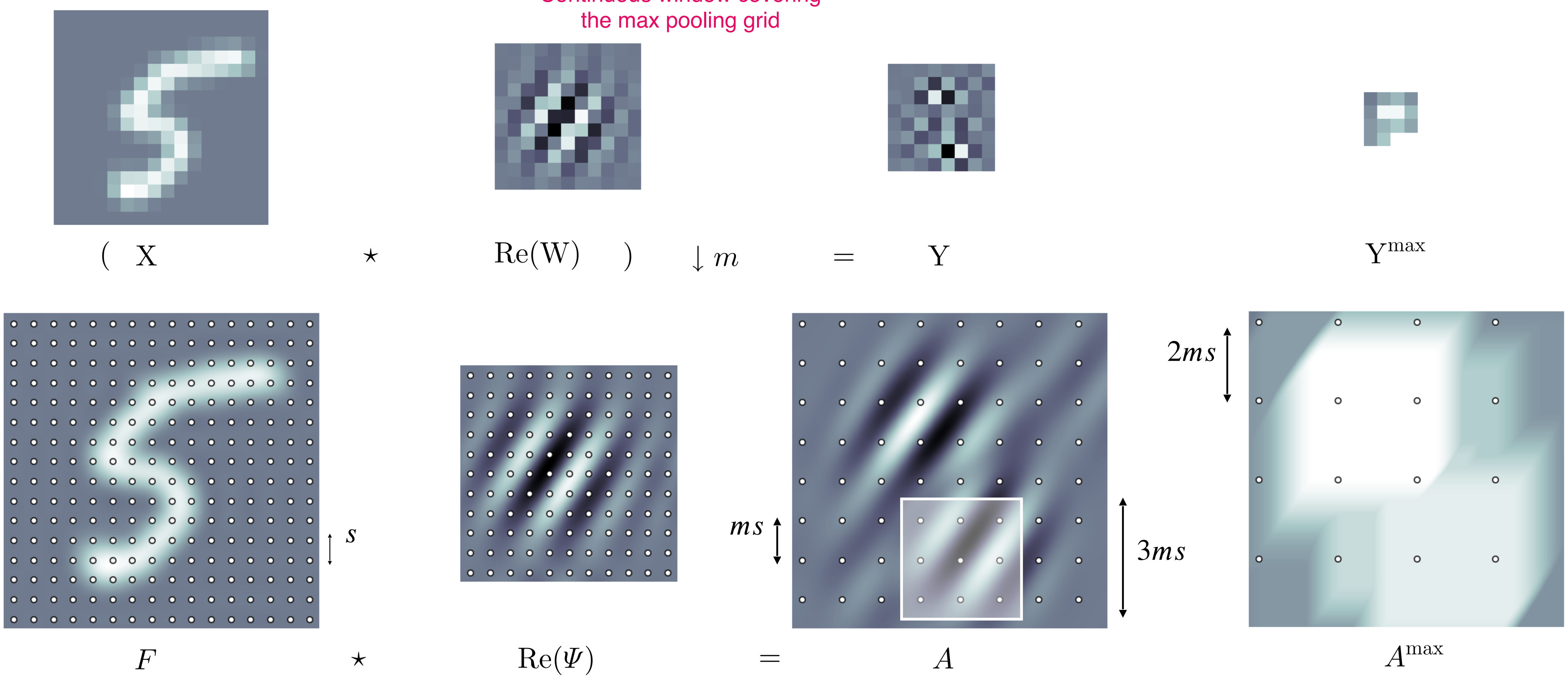
- What if we search for the maximum **continuously** in the window?

$\mathbb{R}\mathbf{Max_0}$

$$\mathrm{Y}_0^{\max}[\boldsymbol{n}] = \max_{\|\boldsymbol{h}\|_\infty \leq \frac{3ms}{2}} \big(F \star \mathrm{Re}(\Psi)\big)(\boldsymbol{x_n} + \boldsymbol{h}) \approx \mathrm{Y}^{\max}[\boldsymbol{n}]?$$

Continuous window covering the max pooling grid

Browsing the whole window



Detour via the continuous framework

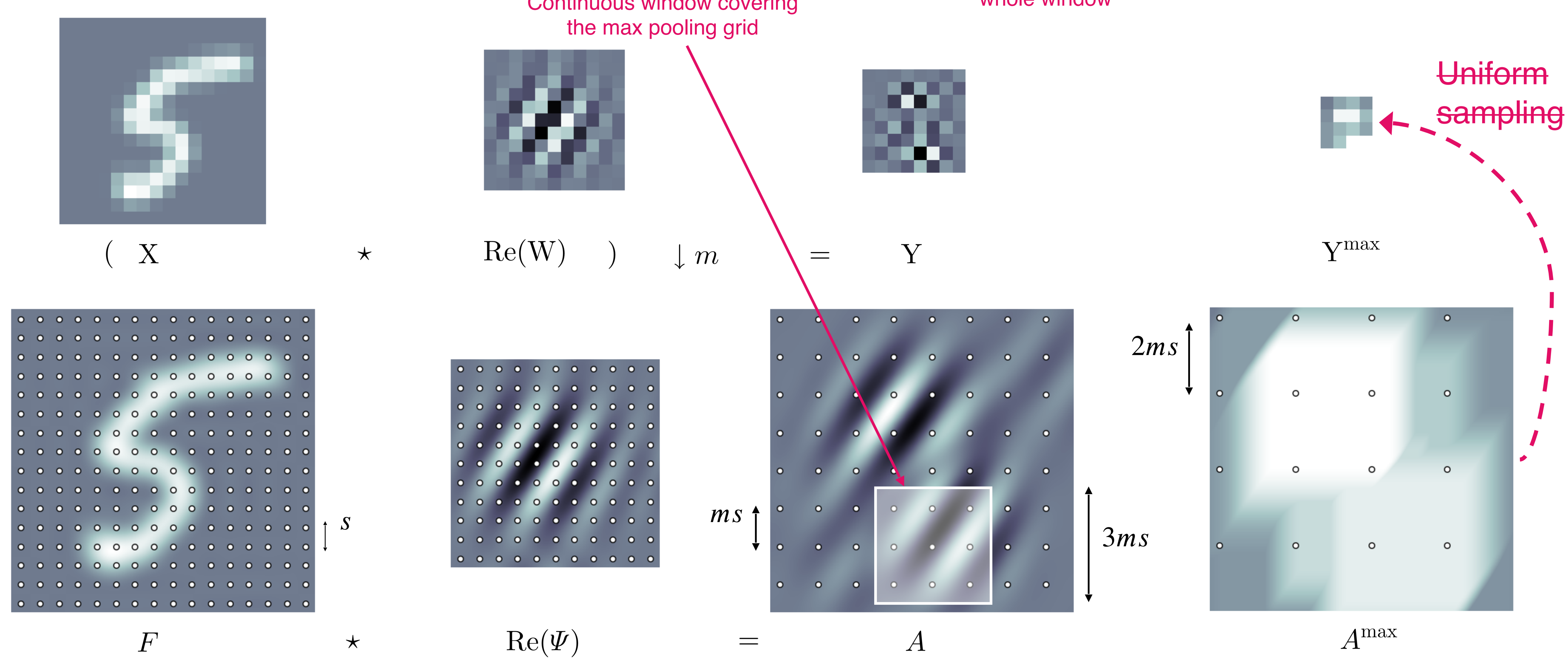
- What if we search for the maximum **continuously** in the window?

$$\mathrm{Y}_0^{\max}[\boldsymbol{n}] = \max_{\|\boldsymbol{h}\|_\infty \le \frac{3ms}{2}} \big(F \star \mathrm{Re}(\Psi)\big)(\boldsymbol{x_n} + \boldsymbol{h}) \approx \mathrm{Y}^{\max}[\boldsymbol{n}]?$$

$\mathbb{R}\mathbf{Max_0}$

Continuous window covering the max pooling grid

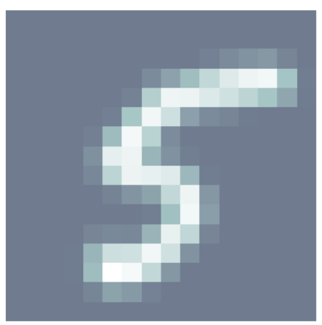
Browsing the whole window

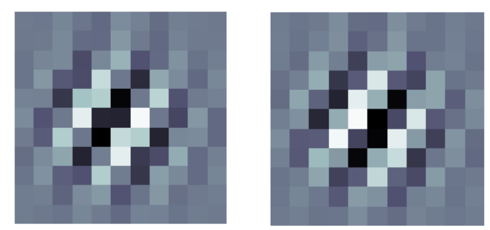


Detour via the continuous framework

- The output $\mathrm{Y}^{\mathrm{mod}}$ can be obtained by a uniform sampling of $|F * \Psi|$

$\mathbb{C}$**Mod**



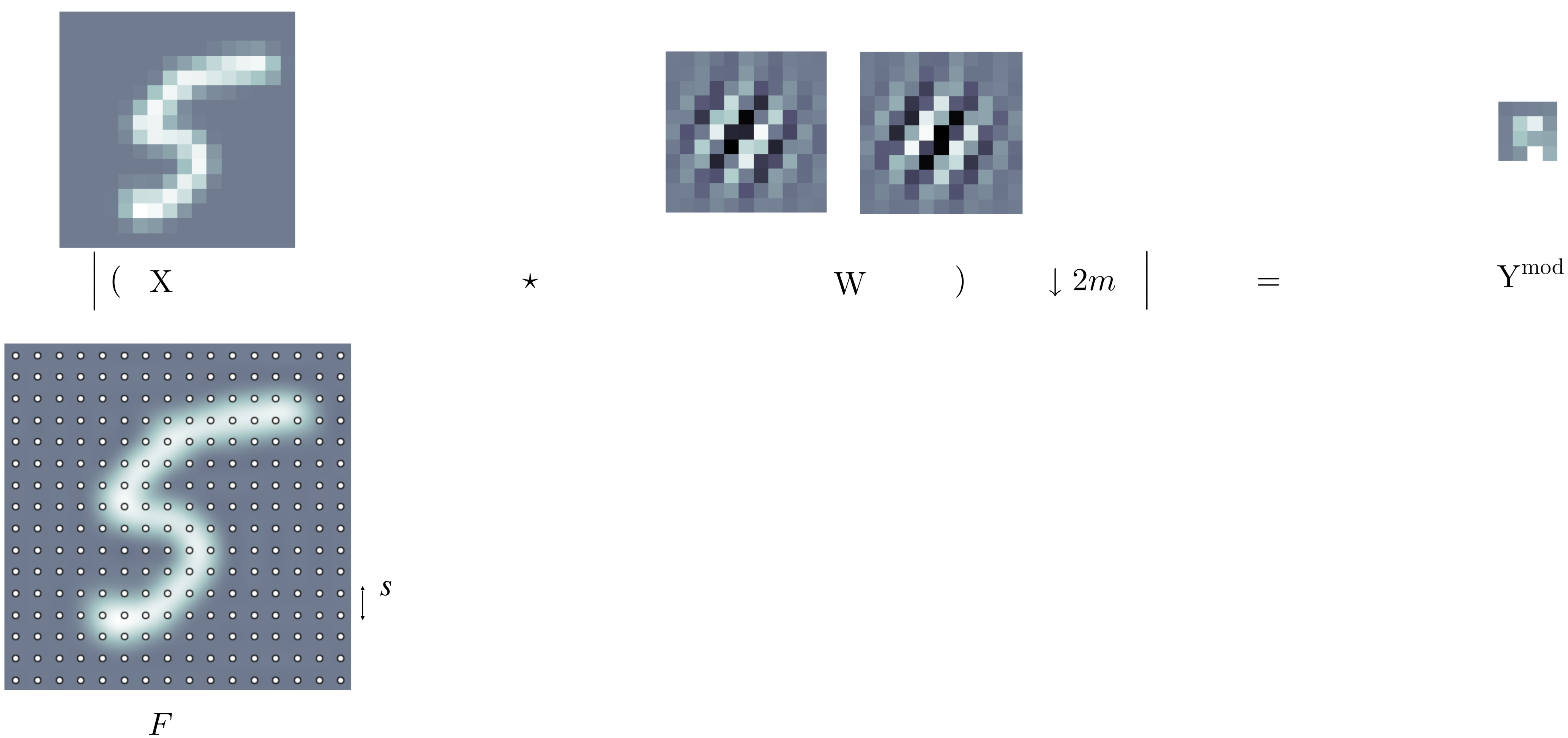


$$\Big| \big(\mathrm{X} \star \mathrm{W} \big) \downarrow 2m \Big| = \mathrm{Y}^{\mathrm{mod}}$$

Detour via the continuous framework

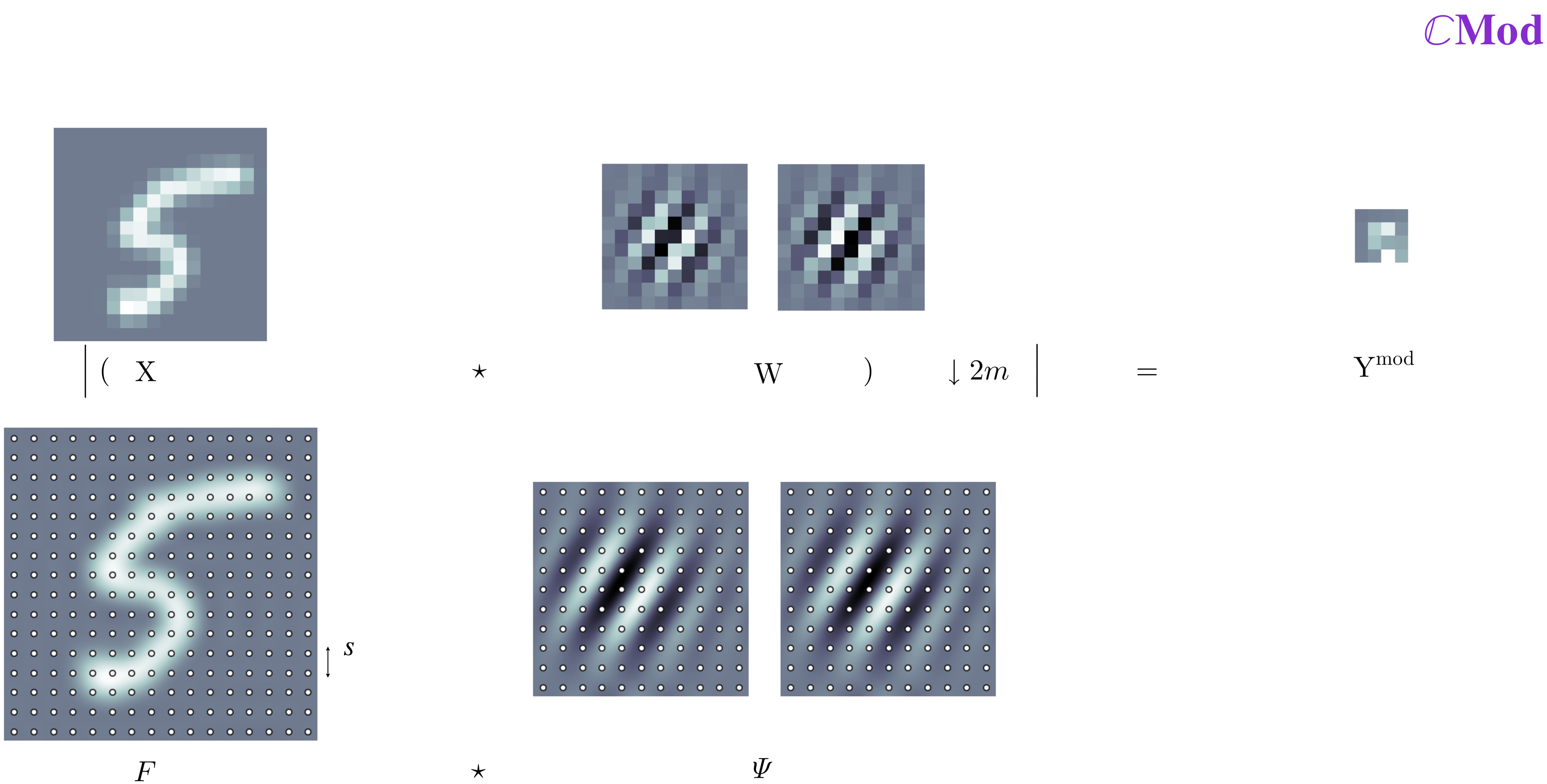
- The output $\mathrm{Y}^{\mathrm{mod}}$ can be obtained by a uniform sampling of $|F * \Psi|$

ℂMod



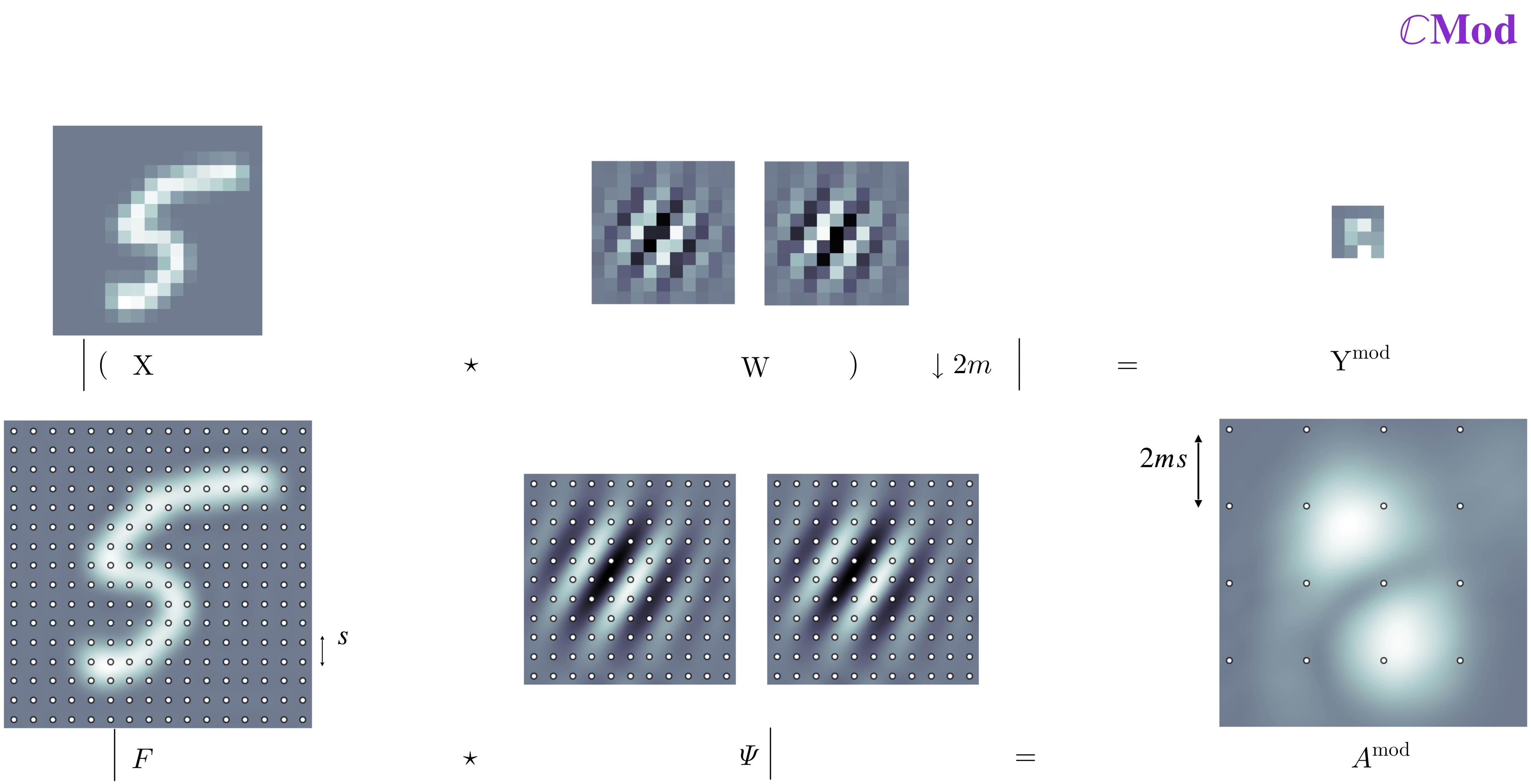
Detour via the continuous framework

- The output $\mathrm{Y}^{\mathrm{mod}}$ can be obtained by a uniform sampling of $|F * \Psi|$



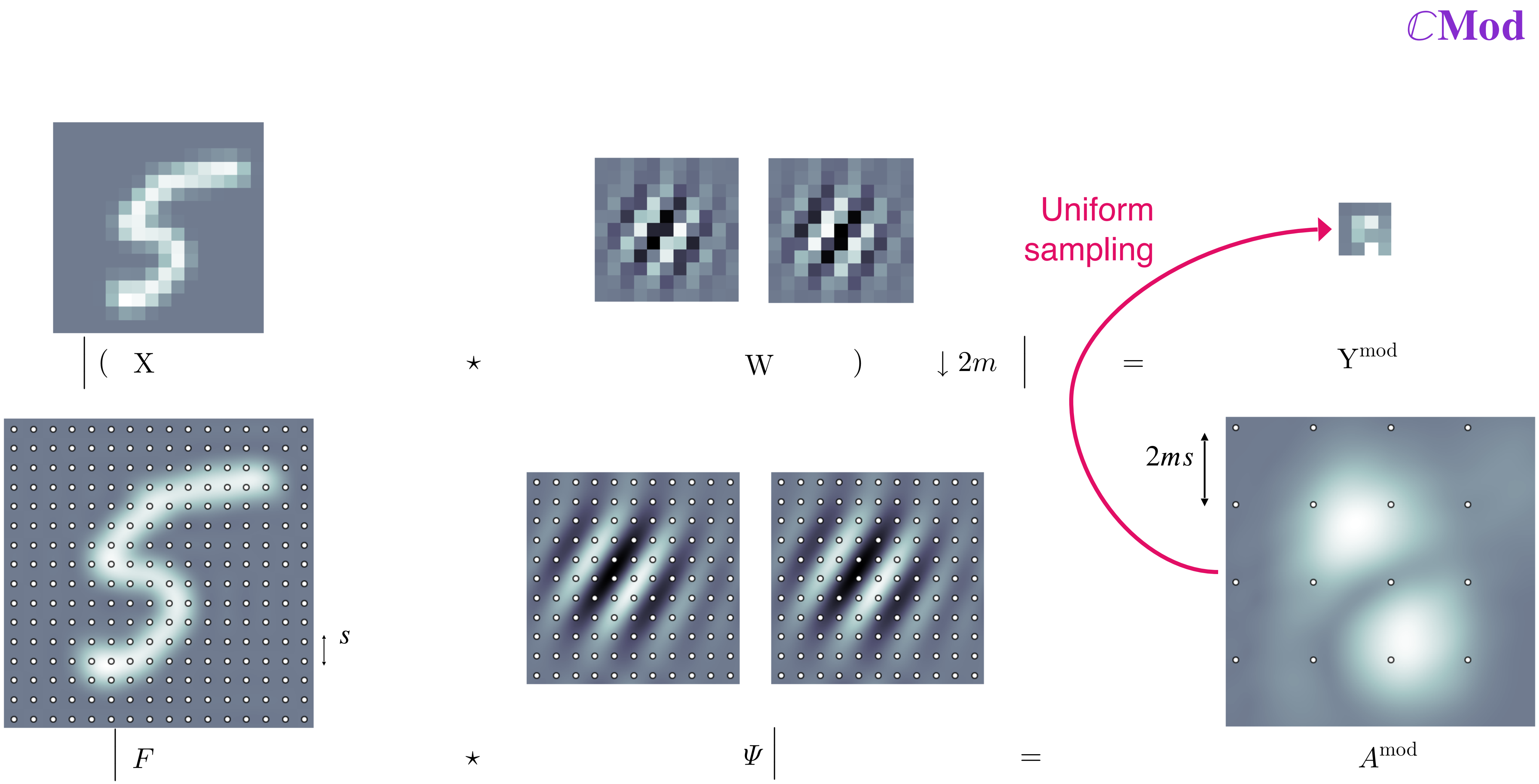
Detour via the continuous framework

- The output $\mathrm{Y}^{\mathrm{mod}}$ can be obtained by a uniform sampling of $|F * \Psi|$



Detour via the continuous framework

- The output $\mathrm{Y}^{\mathrm{mod}}$ can be obtained by a uniform sampling of $|F * \Psi|$

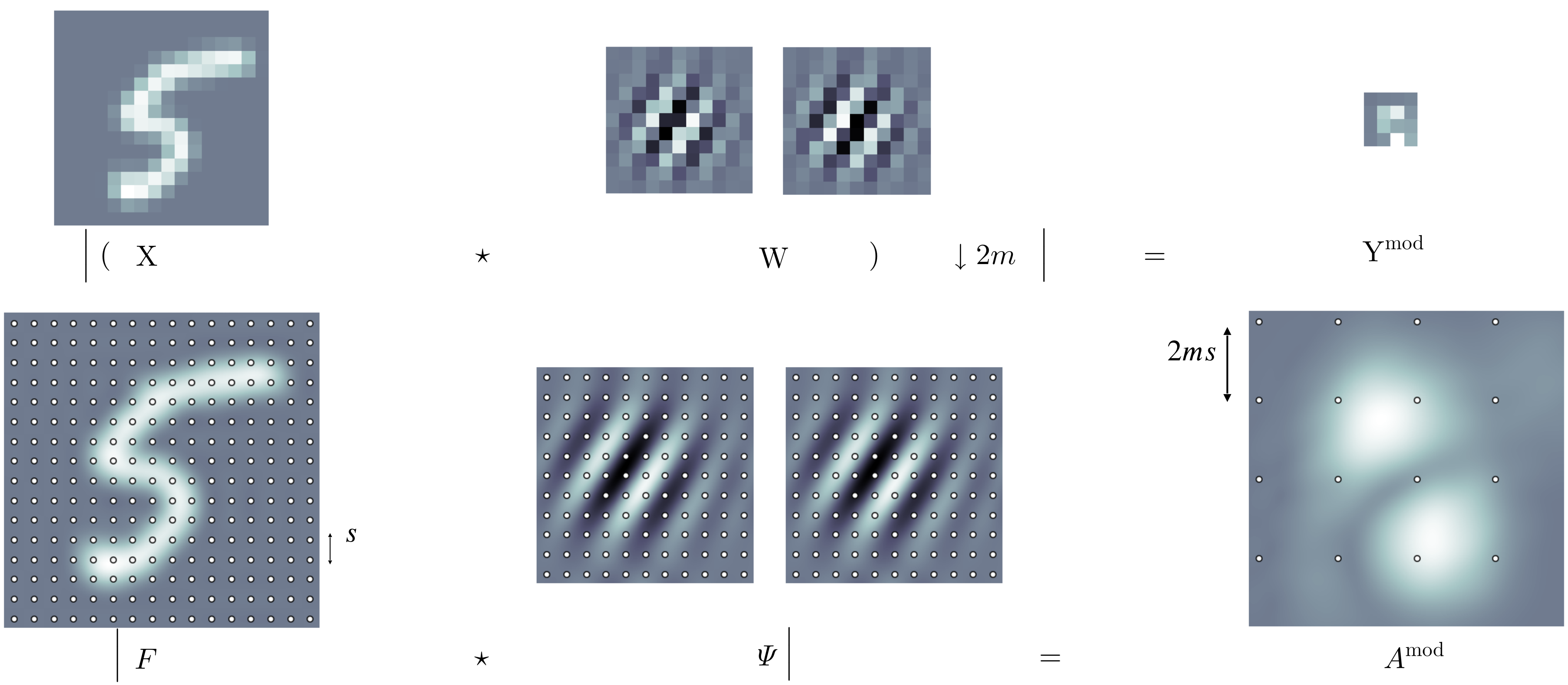


Detour via the continuous framework

- The output $\mathrm{Y}^{\mathrm{mod}}$ can be obtained by a uniform sampling of $|F * \Psi|$

$$\mathrm{Y}^{\mathrm{mod}}[\boldsymbol{n}] = \left|\left(F \star \Psi\right)(2ms\boldsymbol{n})\right|$$

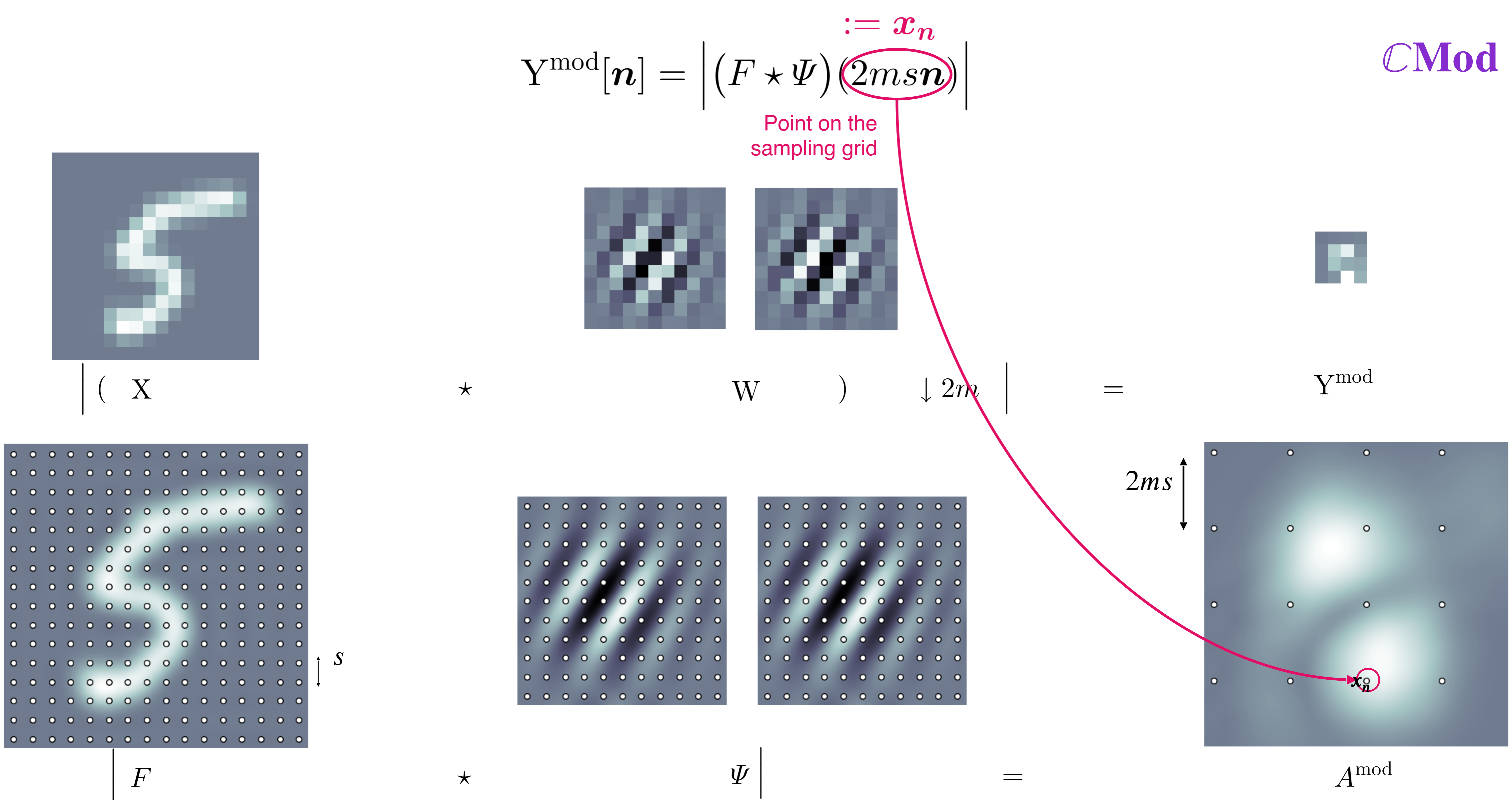
ℂMod



Detour via the continuous framework

- The output $\mathrm{Y}^{\mathrm{mod}}$ can be obtained by a uniform sampling of $|F * \Psi|$

$$\mathrm{Y}^{\mathrm{mod}}[\boldsymbol{n}] = \left|(F \star \Psi)(2ms\boldsymbol{n})\right|$$



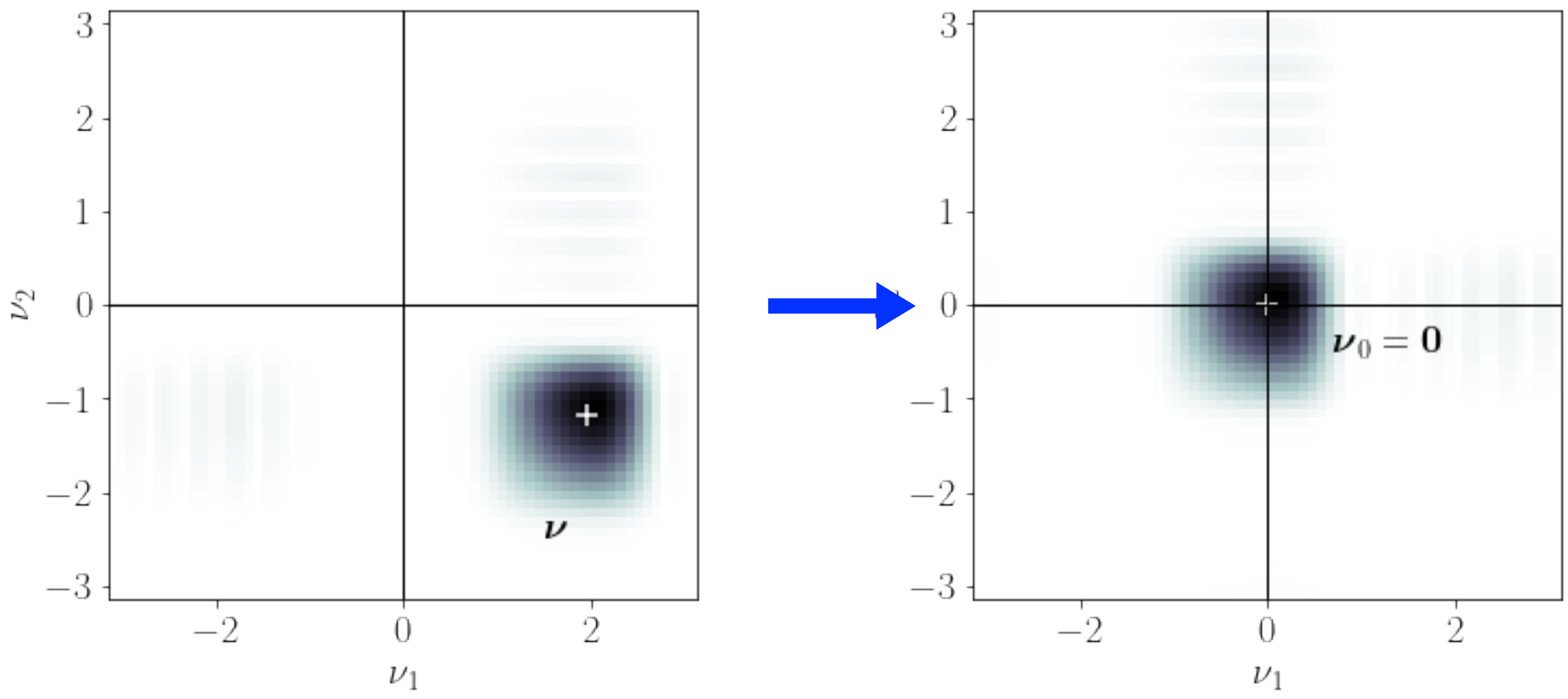
From high to low-frequency

From high to low-frequency

- $\mathcal{V}\big(\boldsymbol{\nu},\, \varepsilon\big) := \Big\{\Psi \in L^2_{\mathbb{C}}(\mathbb{R}^2) \;\Big|\; \operatorname{supp} \widehat{\Psi} \subset B_\infty(\boldsymbol{\nu},\, \varepsilon/2)\Big\}.$

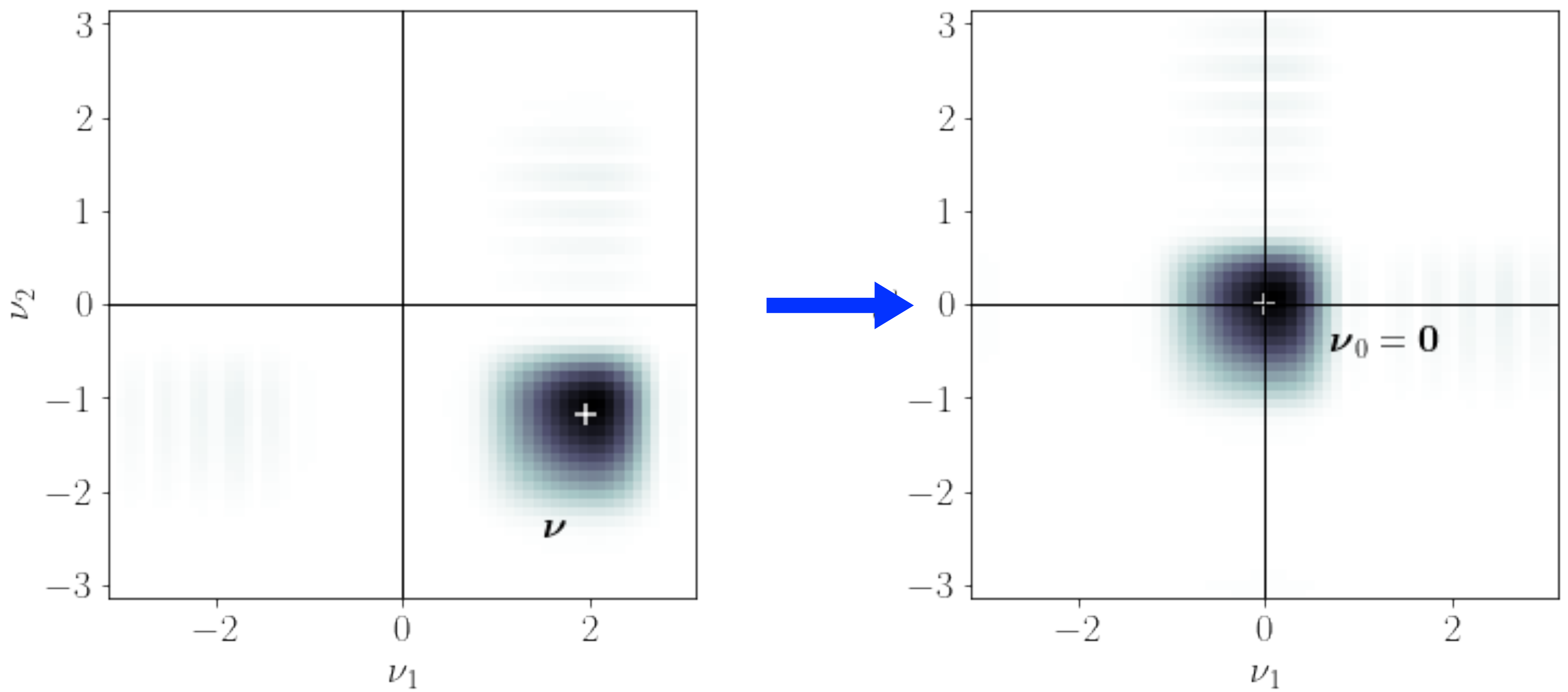
From high to low-frequency

- $\mathcal{V}(\boldsymbol{\nu}, \varepsilon) := \left\{ \Psi \in L^2_{\mathbb{C}}(\mathbb{R}^2) \,\middle|\, \operatorname{supp} \widehat{\Psi} \subset B_\infty(\boldsymbol{\nu}, \varepsilon/2) \right\}.$
- $F_0 : \boldsymbol{x} \mapsto (F * \overline{\Psi})(\boldsymbol{x}) \mathrm{e}^{i\langle \boldsymbol{\nu}, \boldsymbol{x} \rangle} \xrightarrow{\Psi \in \mathcal{V}(\boldsymbol{\nu}, \varepsilon)} \operatorname{supp} \widehat{F_0} \subset B_\infty(\varepsilon/2)$



From high to low-frequency

- $\mathcal{V}(\boldsymbol{\nu},\,\varepsilon) := \left\{ \Psi \in L^2_{\mathbb{C}}(\mathbb{R}^2) \,\middle|\, \operatorname{supp} \widehat{\Psi} \subset B_\infty(\boldsymbol{\nu},\,\varepsilon/2) \right\}.$
- $F_0 : \boldsymbol{x} \mapsto (F * \overline{\Psi})(\boldsymbol{x})\mathrm{e}^{i\langle \boldsymbol{\nu}, \boldsymbol{x}\rangle} \xrightarrow{\Psi \in \mathcal{V}(\boldsymbol{\nu},\varepsilon)} \operatorname{supp}\widehat{F_0} \subset B_\infty(\varepsilon/2)$



- **Shift-invariance bound** for low-frequency functions:

$$\|\mathcal{T}_{\boldsymbol{h}} F_0 - F_0\|_{L^2} \le \alpha(\varepsilon \boldsymbol{h})\|F_0\|_{L^2} \qquad \alpha : \boldsymbol{\tau} \mapsto \frac{\|\boldsymbol{\tau}\|_1}{2}$$

Shift-invariance of CMod in the discrete framework

Shift-invariance of CMod in the discrete framework

- $\mathrm{W} \in \mathcal{J}\big(\boldsymbol{\theta},\, \kappa\big)$

Shift-invariance of CMod in the discrete framework

- $\mathrm{W} \in \mathcal{J}\big(\boldsymbol{\theta},\, \kappa\big)$
- $F_0 : \boldsymbol{x} \mapsto (F_{\mathrm{X}} * \overline{\Psi}_{\mathrm{W}})(\boldsymbol{x})\, \mathrm{e}^{i\langle \boldsymbol{\theta}/s,\, \boldsymbol{x}\rangle} \xrightarrow{\kappa \leq \pi/m} F_0 \in \mathcal{V}(\mathbf{0}, 2\pi/s)$

Shift-invariance of CMod in the discrete framework

- $\mathrm{W} \in \mathcal{J}\big(\boldsymbol{\theta},\, \kappa\big)$
- $F_0 : \boldsymbol{x} \mapsto (F_{\mathrm{X}} * \overline{\Psi}_{\mathrm{W}})(\boldsymbol{x})\, \mathrm{e}^{i\langle \boldsymbol{\theta}/s,\, \boldsymbol{x}\rangle} \xrightarrow{\kappa \leq \pi/m} F_0 \in \mathcal{V}(\mathbf{0}, 2\pi/s)$
- $\displaystyle\sum_{\boldsymbol{n}\in\mathbb{Z}^2} \Big|\mathcal{T}_{\boldsymbol{h}} F_0(s'\boldsymbol{n}) - F_0(s'\boldsymbol{n})\Big|^2 = \frac{1}{s'^2}\, \|\mathcal{T}_{\boldsymbol{h}} F_0 - F_0\|^2_{L^2}$ $\qquad s' := 2ms$

Shift-invariance of CMod in the discrete framework

- $\mathrm{W} \in \mathcal{J}\big(\boldsymbol{\theta},\, \kappa\big)$
- $F_0 : \boldsymbol{x} \mapsto (F_{\mathrm{X}} * \overline{\Psi}_{\mathrm{W}})(\boldsymbol{x})\, \mathrm{e}^{i\langle \boldsymbol{\theta}/s,\, \boldsymbol{x}\rangle} \xrightarrow{\kappa \leq \pi/m} F_0 \in \mathcal{V}(\mathbf{0}, 2\pi/s)$
- $\displaystyle\sum_{\boldsymbol{n}\in\mathbb{Z}^2} \Big|\mathcal{T}_{\boldsymbol{h}} F_0(s'\boldsymbol{n}) - F_0(s'\boldsymbol{n})\Big|^2 = \frac{1}{s'^2} \left\|\mathcal{T}_{\boldsymbol{h}} F_0 - F_0\right\|_{L^2}^2$ $s' := 2ms$
- $\left\|U_m^{\mathrm{mod}}\mathrm{X}\right\|_2 = \dfrac{1}{s'} \left\|F_0\right\|_{L^2}$

Shift-invariance of CMod in the discrete framework

- $\mathrm{W} \in \mathcal{J}(\boldsymbol{\theta},\, \kappa)$
- $F_0 : \boldsymbol{x} \mapsto (F_\mathrm{X} * \overline{\Psi}_\mathrm{W})(\boldsymbol{x})\, \mathrm{e}^{i\langle \boldsymbol{\theta}/s,\, \boldsymbol{x}\rangle} \xrightarrow{\kappa \leq \pi/m} F_0 \in \mathcal{V}(\mathbf{0}, 2\pi/s)$
- $\sum_{\boldsymbol{n}\in\mathbb{Z}^2} \left| \mathcal{T}_{\boldsymbol{h}} F_0(s'\boldsymbol{n}) - F_0(s'\boldsymbol{n}) \right|^2 = \frac{1}{s'^2} \| \mathcal{T}_{\boldsymbol{h}} F_0 - F_0 \|_{L^2}^2 \qquad s' := 2ms$
- $\left\| U_m^{\mathrm{mod}} \mathrm{X} \right\|_2 = \frac{1}{s'} \|F_0\|_{L^2}$

Theorem (Shift invariance of $\mathbb{C}$**Mod**)

If $\mathrm{W} \in \mathcal{J}(\boldsymbol{\theta},\, \kappa)$ and $\kappa \leq \pi/m$

then for any input image with finite support $\mathrm{X} \in l^2_{\mathbb{R}}(\mathbb{Z}^2)$

$$\left\| U_m^{\mathrm{mod}}(\mathcal{T}_{\boldsymbol{u}} \mathrm{X}) - U_m^{\mathrm{mod}} \mathrm{X} \right\|_2 \leq \alpha(\kappa \boldsymbol{u}) \left\| U_m^{\mathrm{mod}} \mathrm{X} \right\|_2$$

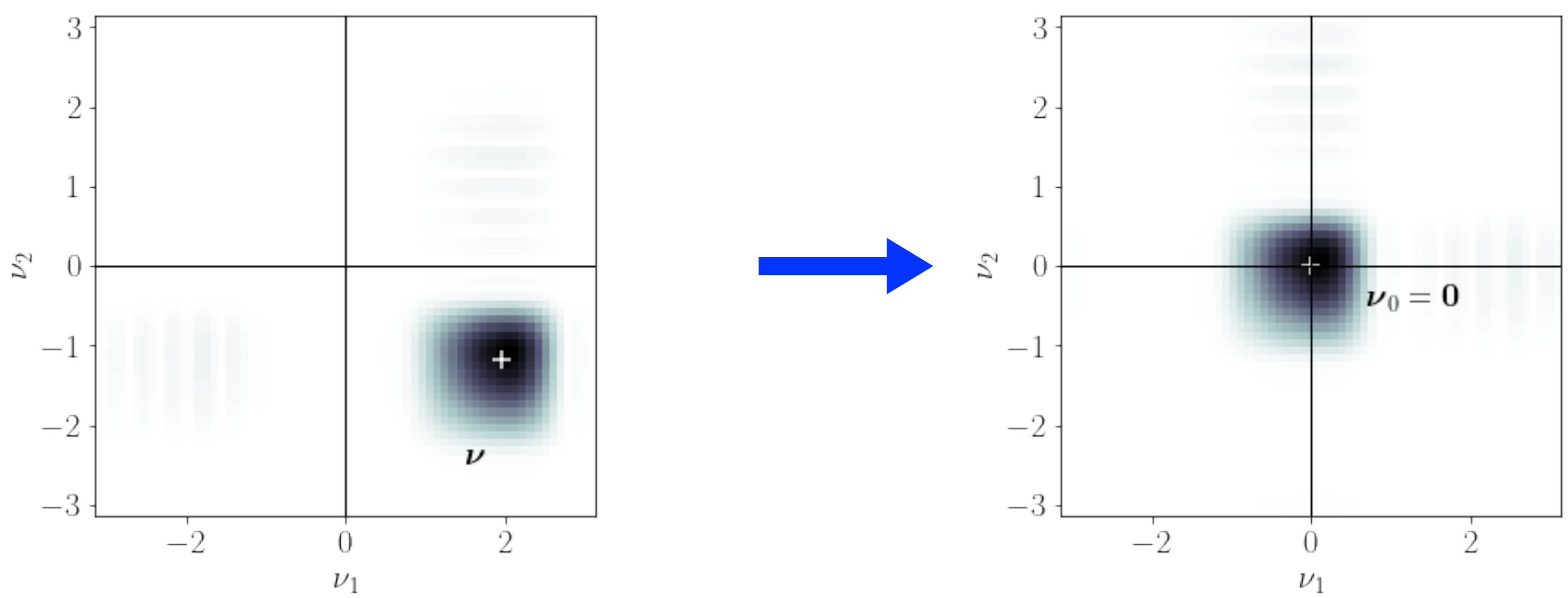
From CMod to RMax in the continuous framework

- **I. Waldspurger intuition** linking the two operators:

From CMod to RMax in the continuous framework

- **I. Waldspurger intuition** linking the two operators:

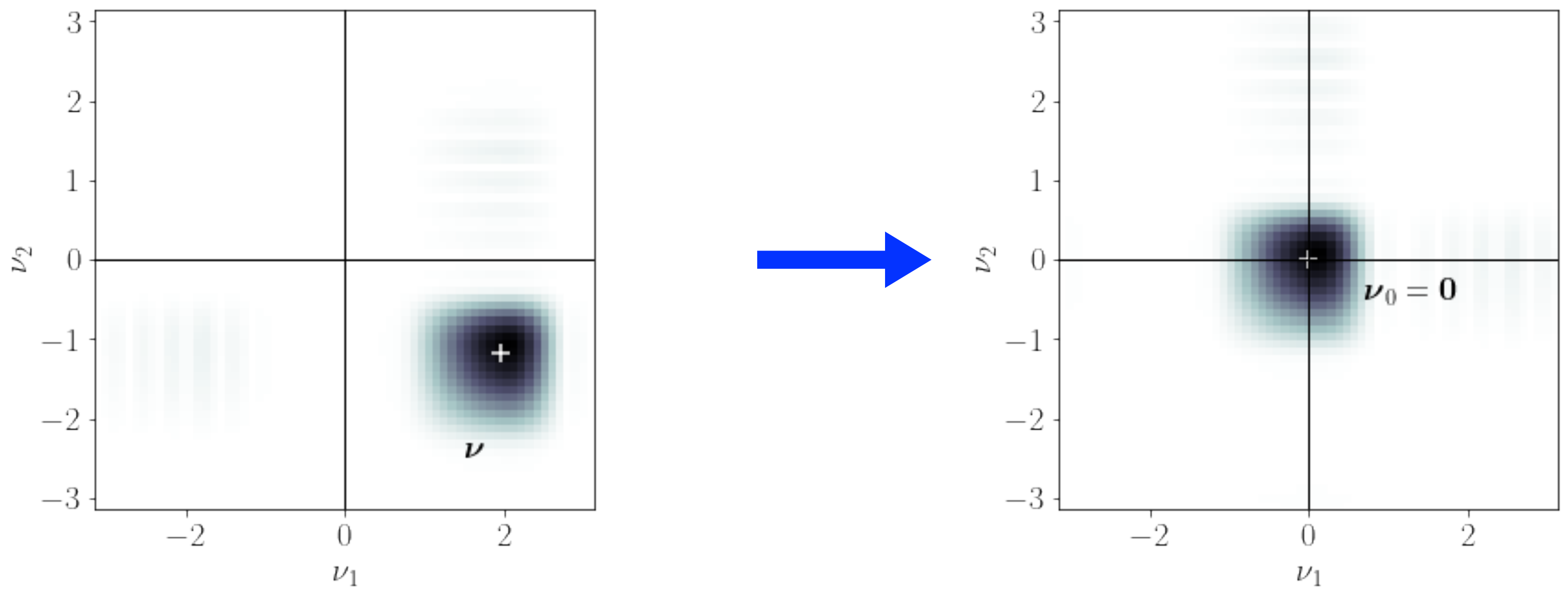
$$F_0 : \boldsymbol{x} \mapsto (F * \overline{\Psi})(\boldsymbol{x}) \mathrm{e}^{i\langle \boldsymbol{\nu}, \boldsymbol{x} \rangle} \xrightarrow{\Psi \in \mathcal{V}(\boldsymbol{\nu}, \varepsilon)} \mathrm{supp}\widehat{F_0} \subset B_\infty(\varepsilon/2)$$



From CMod to RMax in the continuous framework

- **I. Waldspurger intuition** linking the two operators:

$$F_0 : \boldsymbol{x} \mapsto (F * \overline{\Psi})(\boldsymbol{x})\mathrm{e}^{i\langle \boldsymbol{\nu}, \boldsymbol{x}\rangle} \xrightarrow{\Psi \in \mathcal{V}(\boldsymbol{\nu}, \varepsilon)} \mathrm{supp}\widehat{F_0} \subset B_\infty(\varepsilon/2)$$

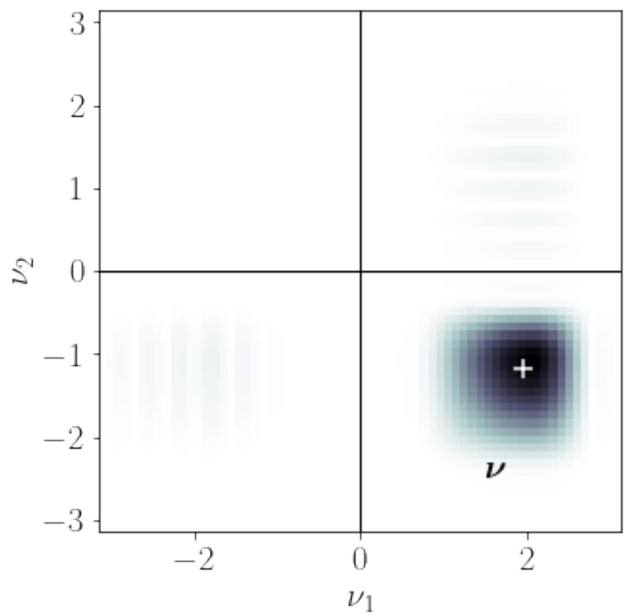


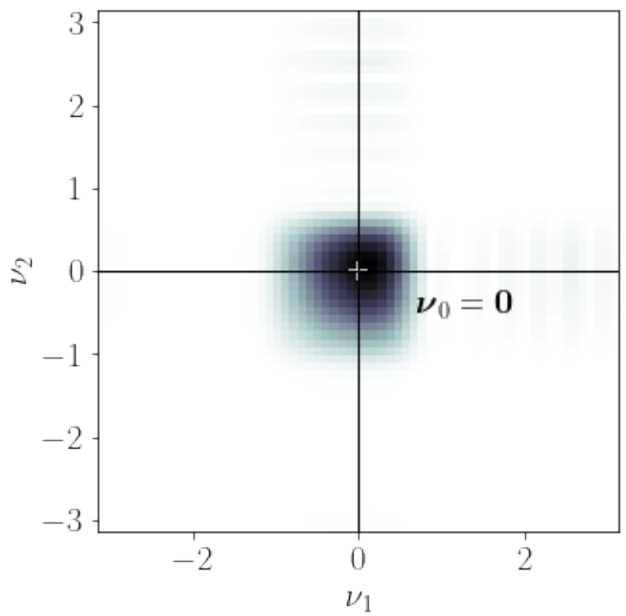
$$(F * \mathrm{Re}\,\overline{\Psi})(\boldsymbol{x}) = \mathrm{Re}((F * \overline{\Psi})(\boldsymbol{x})) = \mathrm{Re}(F_0(\boldsymbol{x})\mathrm{e}^{-i\langle \boldsymbol{\nu}, \boldsymbol{x}\rangle})$$

From CMod to RMax in the continuous framework

- **I. Waldspurger intuition** linking the two operators:

$$F_0 : \boldsymbol{x} \mapsto (F * \overline{\Psi})(\boldsymbol{x})\mathrm{e}^{i\langle \boldsymbol{\nu}, \boldsymbol{x}\rangle} \xrightarrow{\Psi \in \mathcal{V}(\boldsymbol{\nu}, \varepsilon)} \mathrm{supp}\widehat{F_0} \subset B_\infty(\varepsilon/2)$$





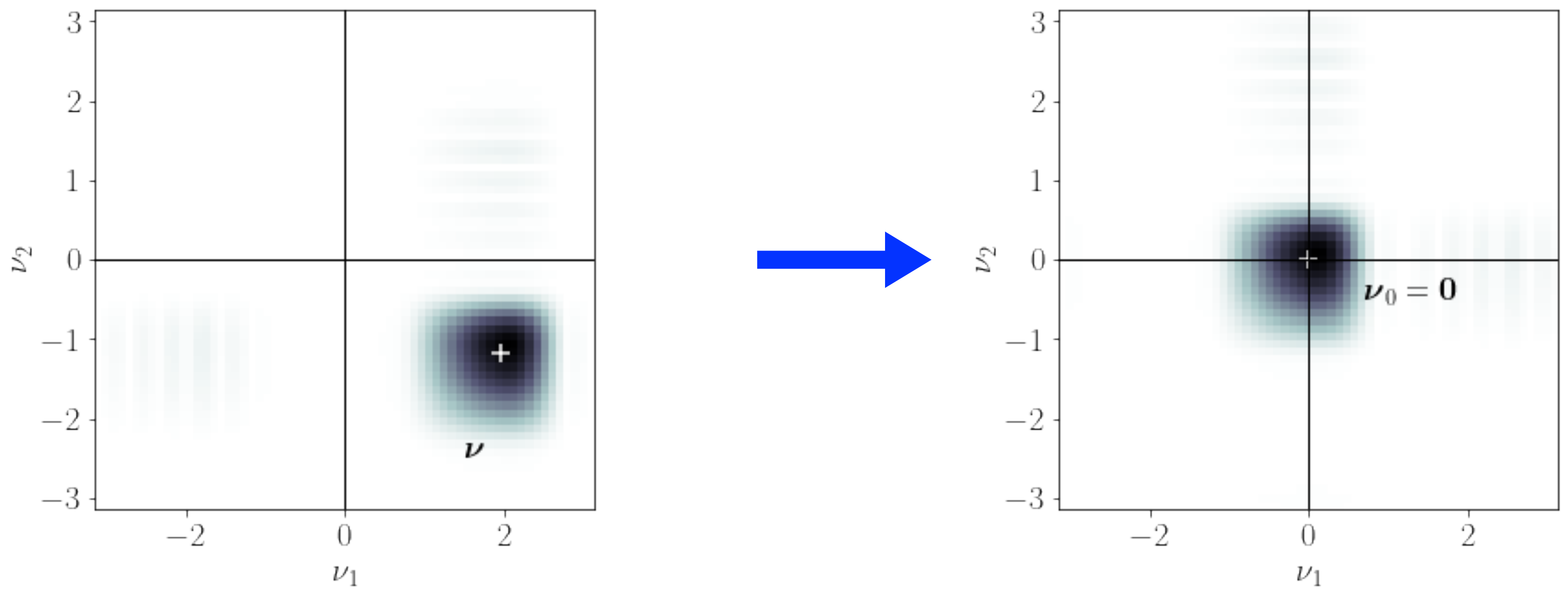
$$(F * \mathrm{Re}\,\overline{\Psi})(\boldsymbol{x}) = \mathrm{Re}((F * \overline{\Psi})(\boldsymbol{x})) = \mathrm{Re}(F_0(\boldsymbol{x})\mathrm{e}^{-i\langle \boldsymbol{\nu}, \boldsymbol{x}\rangle})$$

$$(F * \mathrm{Re}\,\overline{\Psi})(\boldsymbol{x} + \boldsymbol{h}) \approx \mathrm{Re}(F_0(\boldsymbol{x})\mathrm{e}^{-i\langle \boldsymbol{\nu}, \boldsymbol{x}+\boldsymbol{h}\rangle}) = \mathrm{Re}((F * \overline{\Psi})(\boldsymbol{x})\mathrm{e}^{-i\langle \boldsymbol{\nu}, \boldsymbol{h}\rangle})$$

From CMod to RMax in the continuous framework

- **I. Waldspurger intuition** linking the two operators:

$$F_0 : \boldsymbol{x} \mapsto (F * \overline{\Psi})(\boldsymbol{x})\mathrm{e}^{i\langle \boldsymbol{\nu}, \boldsymbol{x}\rangle} \xrightarrow{\Psi \in \mathcal{V}(\boldsymbol{\nu}, \varepsilon)} \mathrm{supp}\widehat{F_0} \subset B_\infty(\varepsilon/2)$$



$$\|\boldsymbol{h}\|_2 \ll 2\pi/\varepsilon \implies F_0(\boldsymbol{x}+\boldsymbol{h}) \approx F_0(\boldsymbol{x})$$

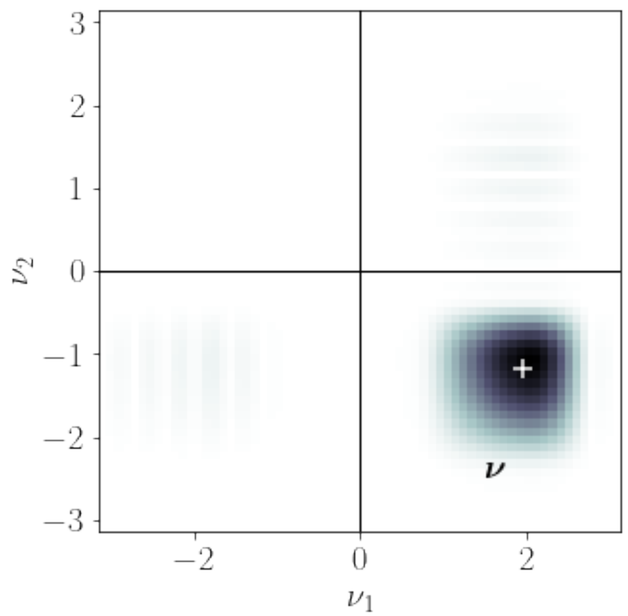
$$(F * \mathrm{Re}\,\overline{\Psi})(\boldsymbol{x}) = \mathrm{Re}((F * \overline{\Psi})(\boldsymbol{x})) = \mathrm{Re}(F_0(\boldsymbol{x})\mathrm{e}^{-i\langle \boldsymbol{\nu}, \boldsymbol{x}\rangle})$$

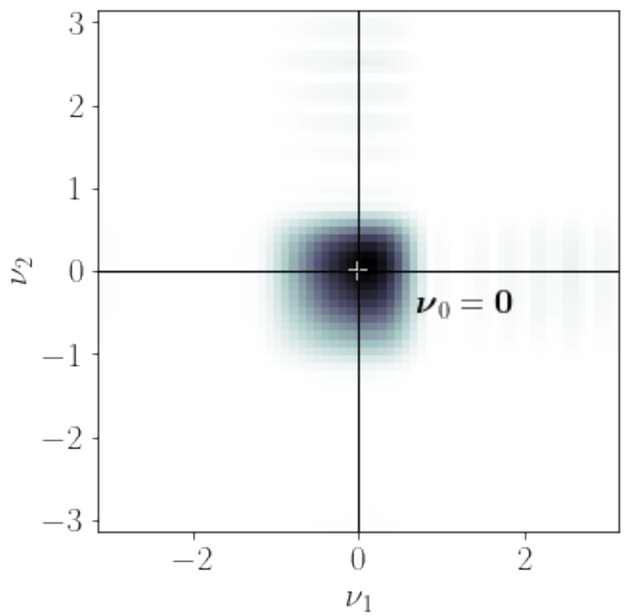
$$(F * \mathrm{Re}\,\overline{\Psi})(\boldsymbol{x}+\boldsymbol{h}) \approx \mathrm{Re}(F_0(\boldsymbol{x})\mathrm{e}^{-i\langle \boldsymbol{\nu}, \boldsymbol{x}+\boldsymbol{h}\rangle}) = \mathrm{Re}((F * \overline{\Psi})(\boldsymbol{x})\mathrm{e}^{-i\langle \boldsymbol{\nu}, \boldsymbol{h}\rangle})$$

From CMod to RMax in the continuous framework

- **I. Waldspurger intuition** linking the two operators:

$$F_0 : \boldsymbol{x} \mapsto (F * \overline{\Psi})(\boldsymbol{x})\mathrm{e}^{i\langle \boldsymbol{\nu}, \boldsymbol{x}\rangle} \xrightarrow{\Psi \in \mathcal{V}(\boldsymbol{\nu}, \varepsilon)} \mathrm{supp}\widehat{F_0} \subset B_\infty(\varepsilon/2)$$





$$(F * \overline{\Psi})(\boldsymbol{x}) = |(F * \overline{\Psi})(\boldsymbol{x})|\mathrm{e}^{-iH(\boldsymbol{x})} \qquad \|\boldsymbol{h}\|_2 \ll 2\pi/\varepsilon \implies F_0(\boldsymbol{x}+\boldsymbol{h}) \approx F_0(\boldsymbol{x})$$

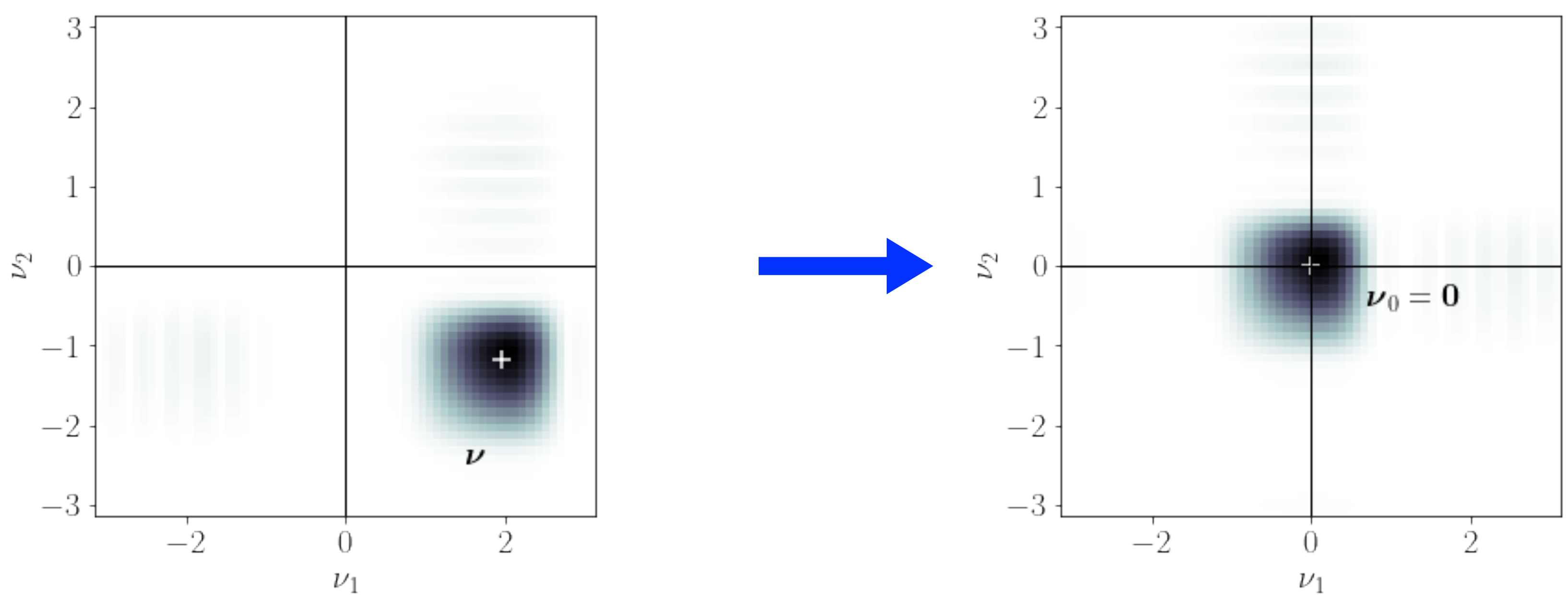
$$(F * \mathrm{Re}\,\overline{\Psi})(\boldsymbol{x}) = \mathrm{Re}((F * \overline{\Psi})(\boldsymbol{x})) = \mathrm{Re}(F_0(\boldsymbol{x})\mathrm{e}^{-i\langle \boldsymbol{\nu}, \boldsymbol{x}\rangle})$$

$$(F * \mathrm{Re}\,\overline{\Psi})(\boldsymbol{x}+\boldsymbol{h}) \approx \mathrm{Re}(F_0(\boldsymbol{x})\mathrm{e}^{-i\langle \boldsymbol{\nu}, \boldsymbol{x}+\boldsymbol{h}\rangle}) = \mathrm{Re}((F * \overline{\Psi})(\boldsymbol{x})\mathrm{e}^{-i\langle \boldsymbol{\nu}, \boldsymbol{h}\rangle})$$

From CMod to RMax in the continuous framework

- **I. Waldspurger intuition** linking the two operators:

$$F_0 : \boldsymbol{x} \mapsto (F * \overline{\Psi})(\boldsymbol{x})\mathrm{e}^{i\langle \boldsymbol{\nu}, \boldsymbol{x}\rangle} \xrightarrow{\Psi \in \mathcal{V}(\boldsymbol{\nu}, \varepsilon)} \mathrm{supp}\widehat{F_0} \subset B_\infty(\varepsilon/2)$$



$$(F * \overline{\Psi})(\boldsymbol{x}) = |(F * \overline{\Psi})(\boldsymbol{x})|\mathrm{e}^{-iH(\boldsymbol{x})} \qquad \|\boldsymbol{h}\|_2 \ll 2\pi/\varepsilon \implies F_0(\boldsymbol{x}+\boldsymbol{h}) \approx F_0(\boldsymbol{x})$$

$$(F * \mathrm{Re}\,\overline{\Psi})(\boldsymbol{x}) = \mathrm{Re}((F * \overline{\Psi})(\boldsymbol{x})) = \mathrm{Re}(F_0(\boldsymbol{x})\mathrm{e}^{-i\langle \boldsymbol{\nu},\, \boldsymbol{x}\rangle})$$

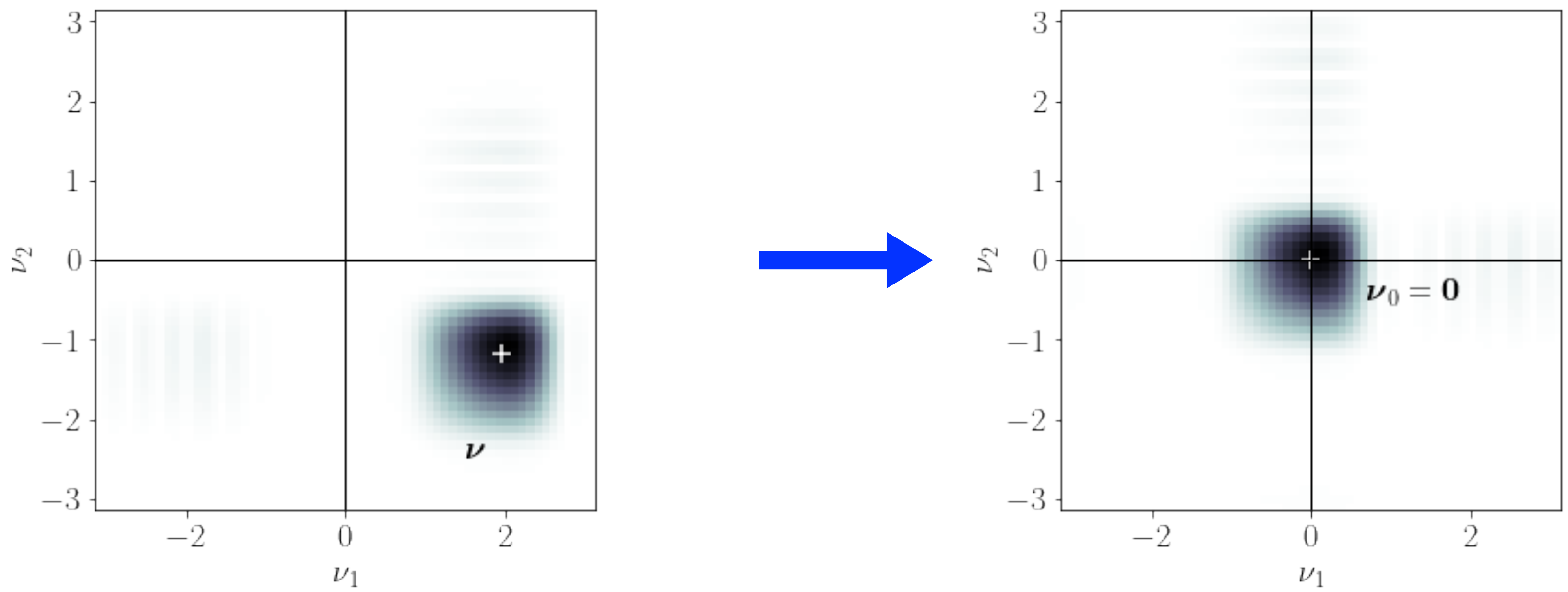
$$(F * \mathrm{Re}\,\overline{\Psi})(\boldsymbol{x}+\boldsymbol{h}) \approx \mathrm{Re}(F_0(\boldsymbol{x})\mathrm{e}^{-i\langle \boldsymbol{\nu},\, \boldsymbol{x}+\boldsymbol{h}\rangle}) = \mathrm{Re}((F * \overline{\Psi})(\boldsymbol{x})\mathrm{e}^{-i\langle \boldsymbol{\nu},\, \boldsymbol{h}\rangle})$$

$$(F * \mathrm{Re}\,\overline{\Psi})(\boldsymbol{x}+\boldsymbol{h}) \approx |(F * \overline{\Psi})(\boldsymbol{x})| \cos(\langle \boldsymbol{\nu},\, \boldsymbol{h}\rangle - H(\boldsymbol{x}))$$

From CMod to RMax in the continuous framework

- **I. Waldspurger intuition** linking the two operators:

$$F_0 : \boldsymbol{x} \mapsto (F * \overline{\Psi})(\boldsymbol{x}) \mathrm{e}^{i\langle \boldsymbol{\nu}, \boldsymbol{x}\rangle} \xrightarrow{\Psi \in \mathcal{V}(\boldsymbol{\nu}, \varepsilon)} \mathrm{supp} \widehat{F_0} \subset B_\infty(\varepsilon/2)$$



$$(F * \overline{\Psi})(\boldsymbol{x}) = |(F * \overline{\Psi})(\boldsymbol{x})| \mathrm{e}^{-iH(\boldsymbol{x})} \qquad \|\boldsymbol{h}\|_2 \ll 2\pi/\varepsilon \implies F_0(\boldsymbol{x} + \boldsymbol{h}) \approx F_0(\boldsymbol{x})$$

$$(F * \mathrm{Re}\, \overline{\Psi})(\boldsymbol{x}) = \mathrm{Re}((F * \overline{\Psi})(\boldsymbol{x})) = \mathrm{Re}(F_0(\boldsymbol{x}) \mathrm{e}^{-i\langle \boldsymbol{\nu}, \, \boldsymbol{x}\rangle})$$

$$(F * \mathrm{Re}\, \overline{\Psi})(\boldsymbol{x} + \boldsymbol{h}) \approx \mathrm{Re}(F_0(\boldsymbol{x}) \mathrm{e}^{-i\langle \boldsymbol{\nu}, \, \boldsymbol{x}+\boldsymbol{h}\rangle}) = \mathrm{Re}((F * \overline{\Psi})(\boldsymbol{x}) \mathrm{e}^{-i\langle \boldsymbol{\nu}, \, \boldsymbol{h}\rangle})$$

$$(F * \mathrm{Re}\, \overline{\Psi})(\boldsymbol{x} + \boldsymbol{h}) \approx |(F * \overline{\Psi})(\boldsymbol{x})| \underbrace{\cos(\langle \boldsymbol{\nu}, \, \boldsymbol{h}\rangle - H(\boldsymbol{x}))}_{G(\boldsymbol{x}, \, \boldsymbol{h})}$$

From CMod to RMax in the continuous framework

- **I. Waldspurger intuition** linking the two operators:

$$(F * \mathrm{Re}\,\overline{\Psi})(\boldsymbol{x}+\boldsymbol{h}) \approx |(F * \overline{\Psi})(\boldsymbol{x})| \cos(\langle \boldsymbol{\nu}, \boldsymbol{h}\rangle - H(\boldsymbol{x}))$$

From CMod to RMax in the continuous framework

- **I. Waldspurger intuition** linking the two operators:

$$(F * \mathrm{Re}\,\overline{\Psi})(\boldsymbol{x}+\boldsymbol{h}) \approx |(F * \overline{\Psi})(\boldsymbol{x})| \cos(\langle \boldsymbol{\nu}, \boldsymbol{h}\rangle - H(\boldsymbol{x}))$$

  - $U^{\mathrm{mod}}[\Psi]\,(F) : \boldsymbol{x} \mapsto \left|(F * \overline{\Psi})(\boldsymbol{x})\right|$
  - $U_r^{\max}[\Psi]\,(F) : \boldsymbol{x} \mapsto \max\limits_{\|\boldsymbol{h}\|_\infty \leq r} (F * \mathrm{Re}\overline{\Psi})(\boldsymbol{x}+\boldsymbol{h})$

From CMod to RMax in the continuous framework

- **I. Waldspurger intuition** linking the two operators:

$$(F * \mathrm{Re}\,\overline{\Psi})(\boldsymbol{x}+\boldsymbol{h}) \approx |(F * \overline{\Psi})(\boldsymbol{x})| \cos(\langle \boldsymbol{\nu}, \boldsymbol{h}\rangle - H(\boldsymbol{x}))$$

- $U^{\mathrm{mod}}[\Psi]\,(F) : \boldsymbol{x} \mapsto \left|(F * \overline{\Psi})(\boldsymbol{x})\right|$
- $U_r^{\max}[\Psi]\,(F) : \boldsymbol{x} \mapsto \max_{\|\boldsymbol{h}\|_\infty \leq r} (F * \mathrm{Re}\overline{\Psi})(\boldsymbol{x}+\boldsymbol{h})$

$$r \ll 2\pi/\varepsilon \implies U_r^{\max} F(\boldsymbol{x}) \approx U^{\mathrm{mod}} F(\boldsymbol{x}) \max_{\|\boldsymbol{h}\|_\infty \leq r} G(\boldsymbol{x}, \boldsymbol{h})$$

Max

From CMod to RMax in the continuous framework

- **I. Waldspurger intuition** linking the two operators:

$$(F * \mathrm{Re}\,\overline{\Psi})(\boldsymbol{x}+\boldsymbol{h}) \approx |(F * \overline{\Psi})(\boldsymbol{x})| \cos(\langle \boldsymbol{\nu}, \boldsymbol{h}\rangle - H(\boldsymbol{x}))$$

- $U^{\mathrm{mod}}[\Psi]\,(F) : \boldsymbol{x} \mapsto \left|(F * \overline{\Psi})(\boldsymbol{x})\right|$
- $U_r^{\max}[\Psi]\,(F) : \boldsymbol{x} \mapsto \max_{\|\boldsymbol{h}\|_\infty \leq r} (F * \mathrm{Re}\overline{\Psi})(\boldsymbol{x}+\boldsymbol{h})$

Max

$$r \ll 2\pi/\varepsilon \implies U_r^{\max} F(\boldsymbol{x}) \approx U^{\mathrm{mod}} F(\boldsymbol{x}) \max_{\|\boldsymbol{h}\|_\infty \leq r} G(\boldsymbol{x}, \boldsymbol{h})$$

- For $r \geq \dfrac{\pi}{\|\boldsymbol{\nu}\|_2}$ the cosine $\boldsymbol{h} \mapsto G(\boldsymbol{x}, \boldsymbol{h})$ reaches 1 on $B_\infty(r)$

From CMod to RMax in the continuous framework

- **I. Waldspurger intuition** linking the two operators:

$$(F * \mathrm{Re}\,\overline{\Psi})(\boldsymbol{x}+\boldsymbol{h}) \approx |(F * \overline{\Psi})(\boldsymbol{x})| \cos(\langle \boldsymbol{\nu},\, \boldsymbol{h}\rangle - H(\boldsymbol{x}))$$

- $U^{\mathrm{mod}}[\Psi]\,(F) : \boldsymbol{x} \mapsto \left|(F * \overline{\Psi})(\boldsymbol{x})\right|$
- $U_r^{\max}[\Psi]\,(F) : \boldsymbol{x} \mapsto \max\limits_{\|\boldsymbol{h}\|_\infty \leq r} (F * \mathrm{Re}\overline{\Psi})(\boldsymbol{x}+\boldsymbol{h})$

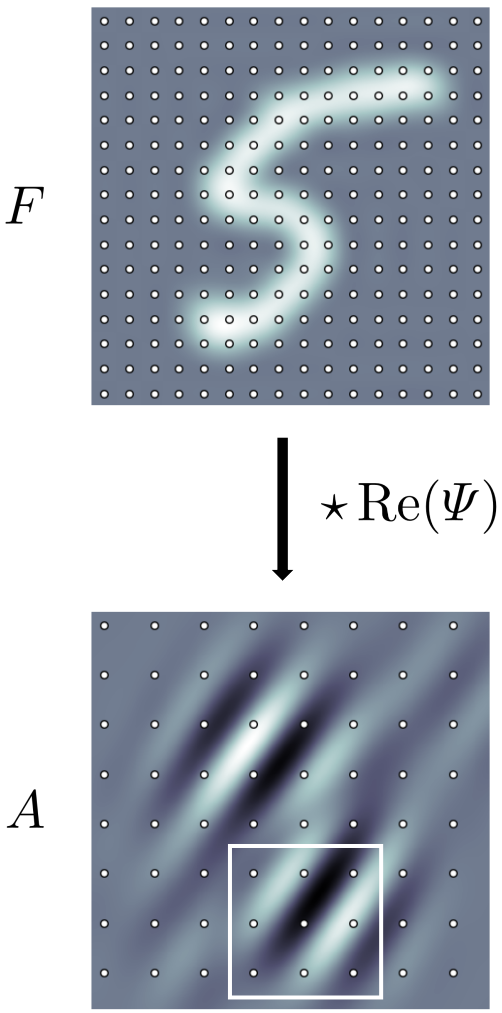
Max

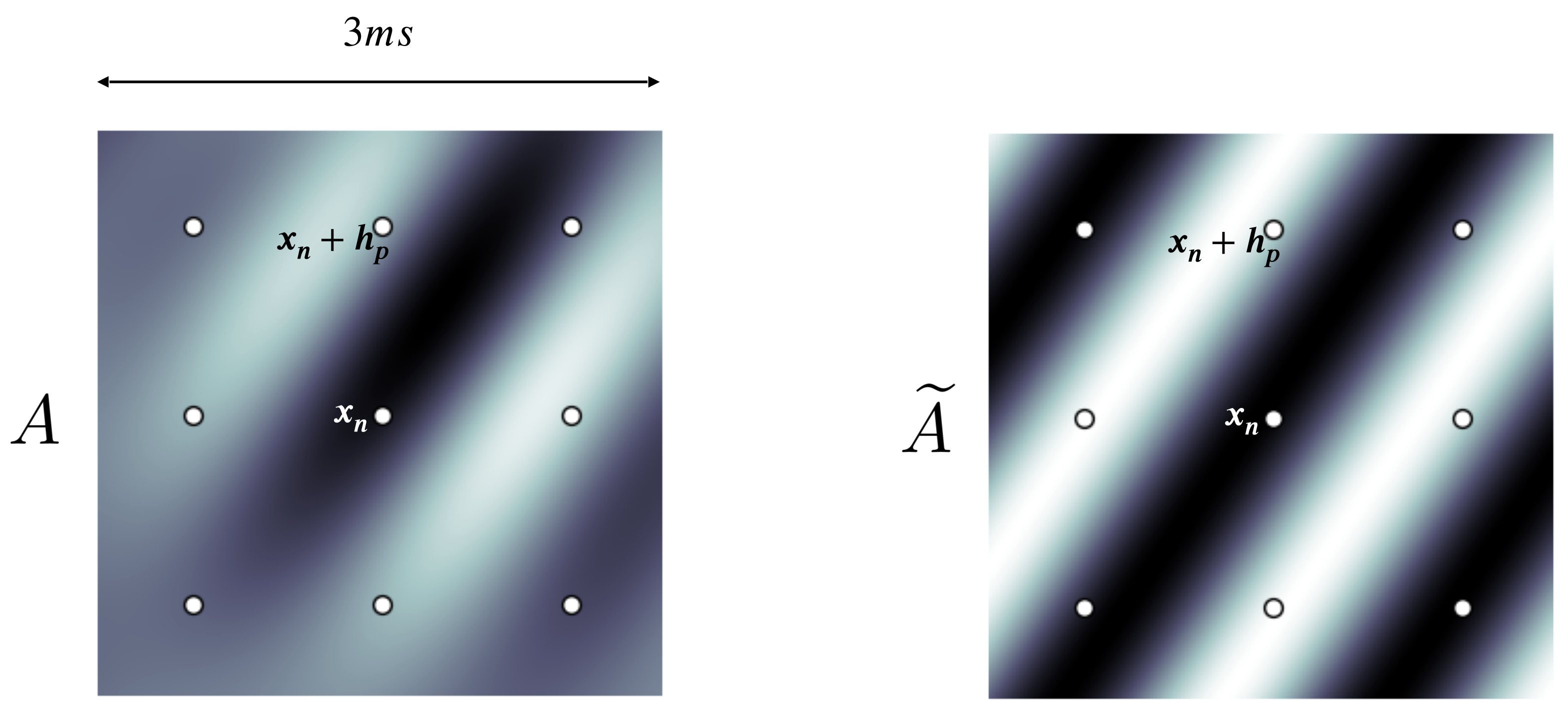
$$r \ll 2\pi/\varepsilon \implies U_r^{\max} F(\boldsymbol{x}) \approx U^{\mathrm{mod}} F(\boldsymbol{x}) \max_{\|\boldsymbol{h}\|_\infty \leq r} G(\boldsymbol{x}, \boldsymbol{h})$$

- For $r \geq \dfrac{\pi}{\|\boldsymbol{\nu}\|_2}$ the cosine $\boldsymbol{h} \mapsto G(\boldsymbol{x}, \boldsymbol{h})$ reaches 1 on $B_\infty(r)$

$$\frac{\pi}{\|\boldsymbol{\nu}\|_2} \leq r \ll \frac{2\pi}{\varepsilon} \implies U_r^{\max} F(\boldsymbol{x}) \approx U^{\mathrm{mod}} F(\boldsymbol{x})$$

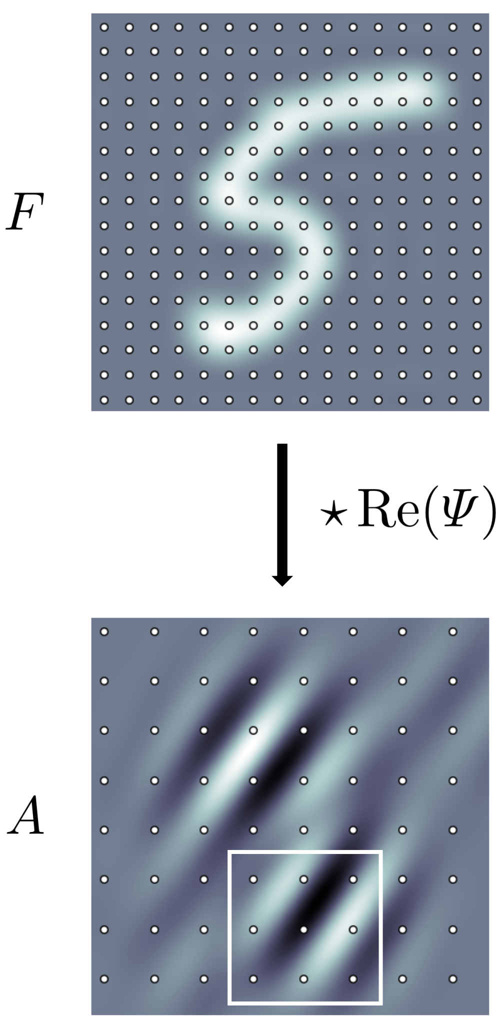
Adaptation to the discrete case

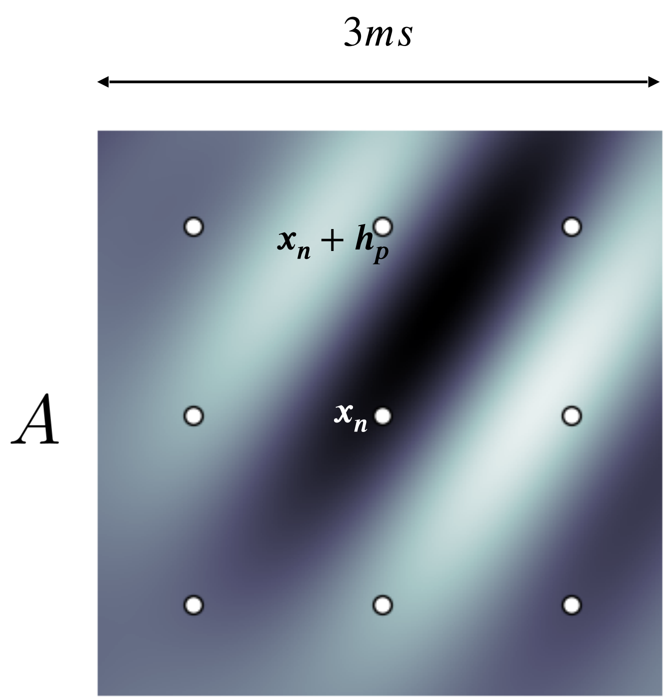


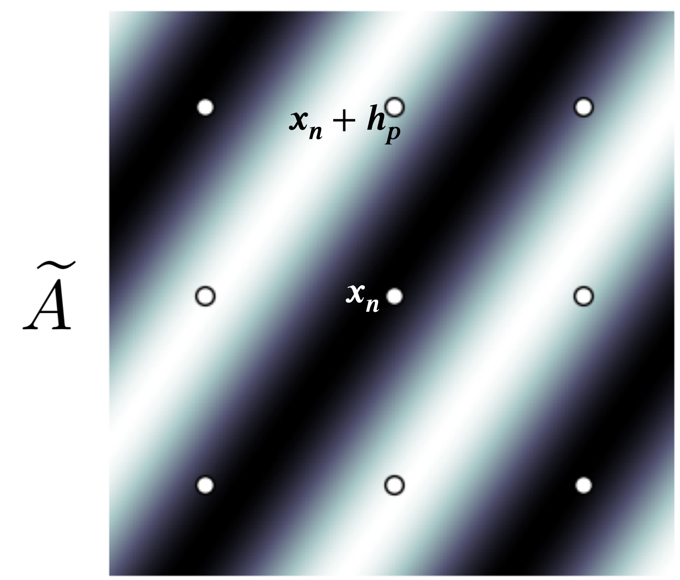


$$\big(F \star \mathrm{Re}(\Psi)\big)(\boldsymbol{x_n} + \boldsymbol{h_p}) \quad \approx \quad |(F \star \Psi)(\boldsymbol{x_n})| \, \cos\big(\langle \boldsymbol{\nu}, \, \boldsymbol{h_p} \rangle - H(\boldsymbol{x_n})\big)$$

Adaptation to the discrete case



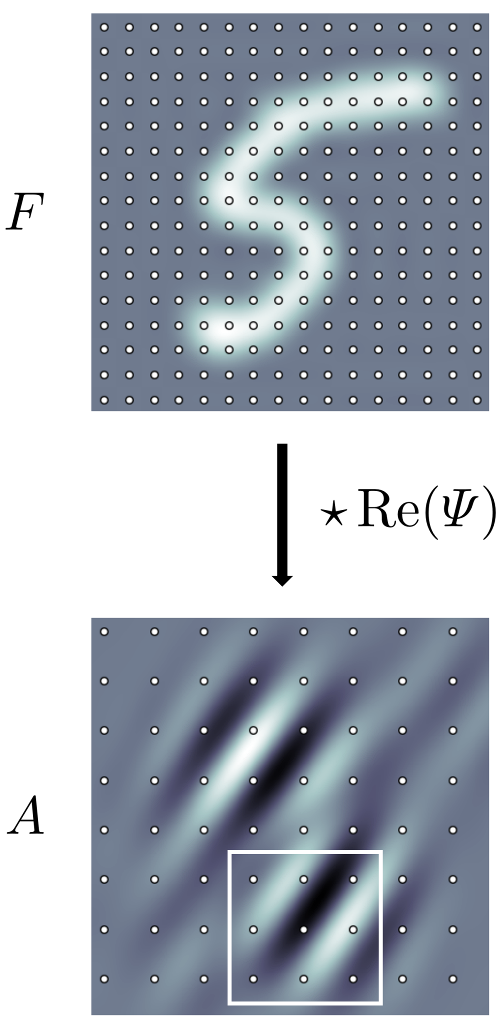


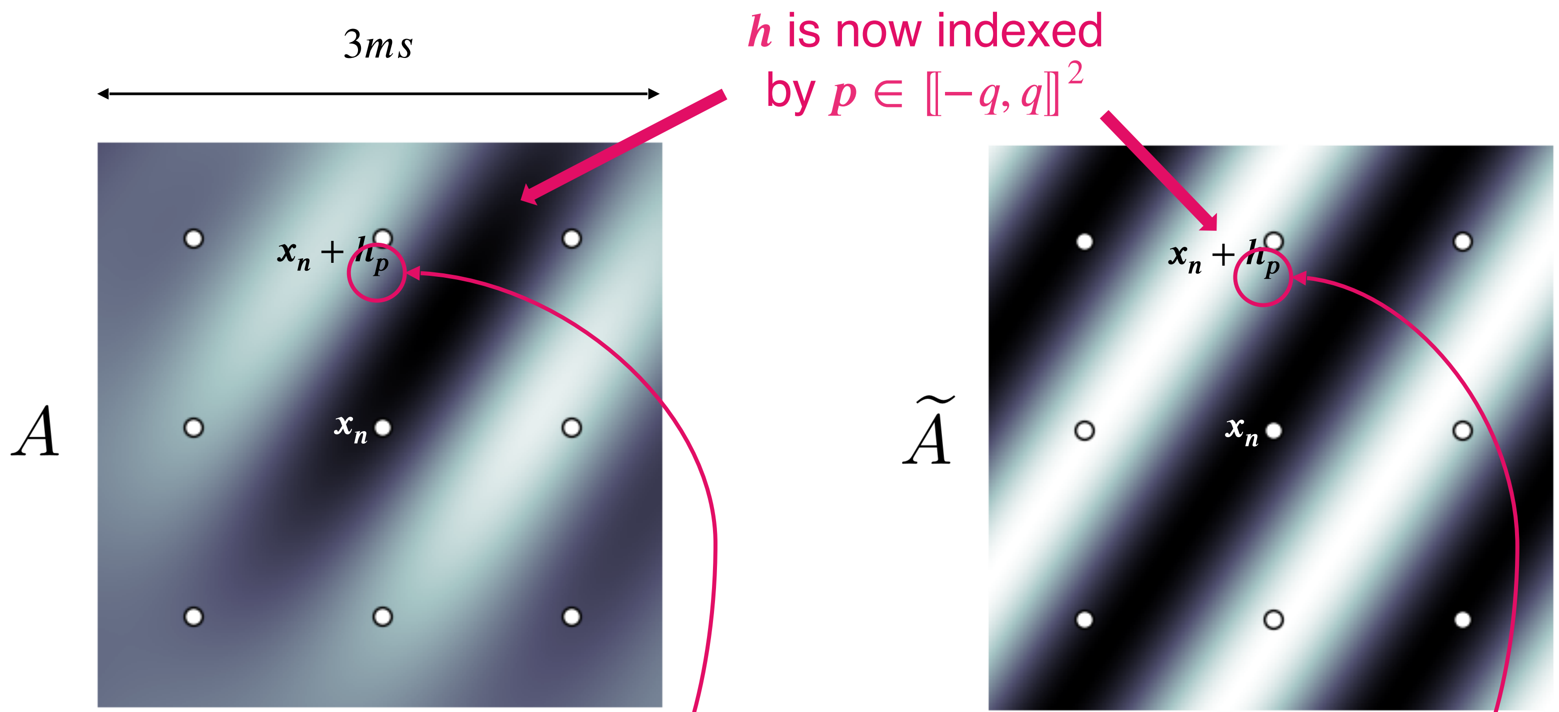


$$\big(F \star \mathrm{Re}(\Psi)\big)(\boldsymbol{x_n} + \boldsymbol{h_p}) \approx \big|(F \star \Psi)(\boldsymbol{x_n})\big| \cos\big(\langle \boldsymbol{\nu},\, \boldsymbol{h_p} \rangle - H(\boldsymbol{x_n})\big)$$

$\mathrm{Y}^{\mathrm{mod}}[\boldsymbol{n}]$

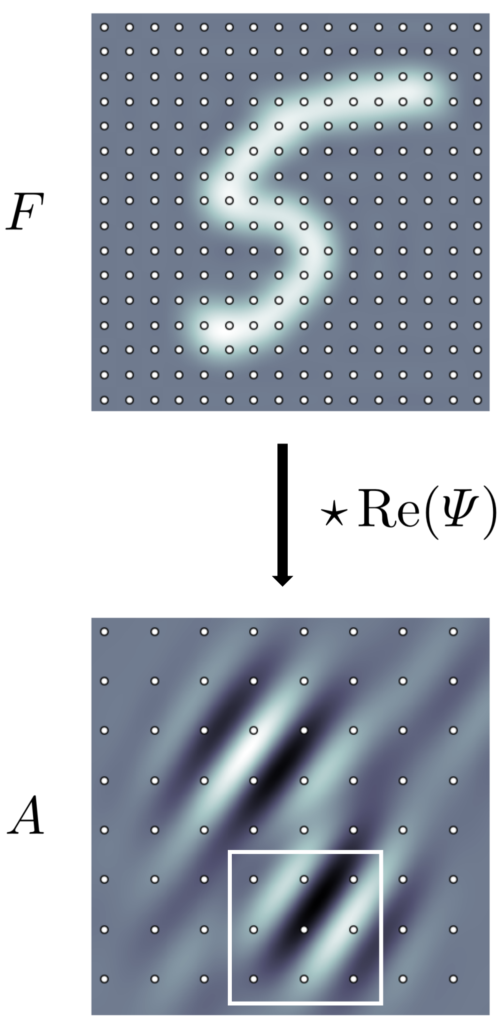
Adaptation to the discrete case

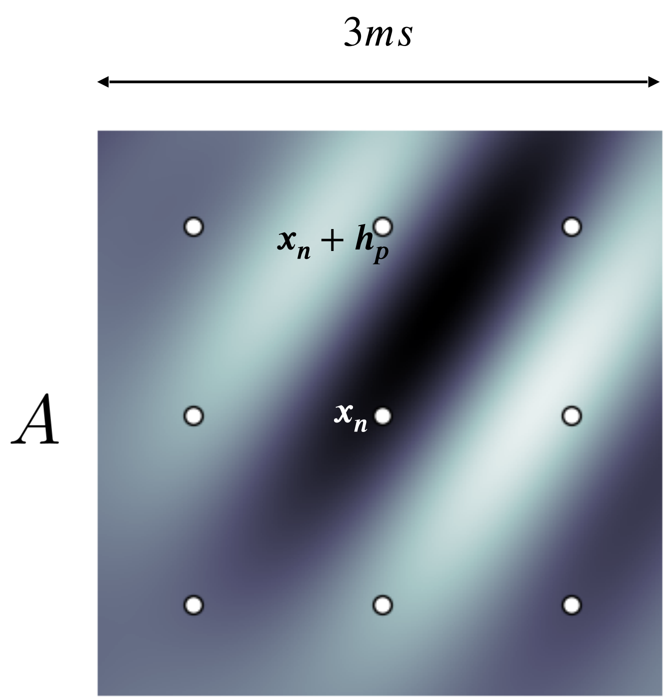


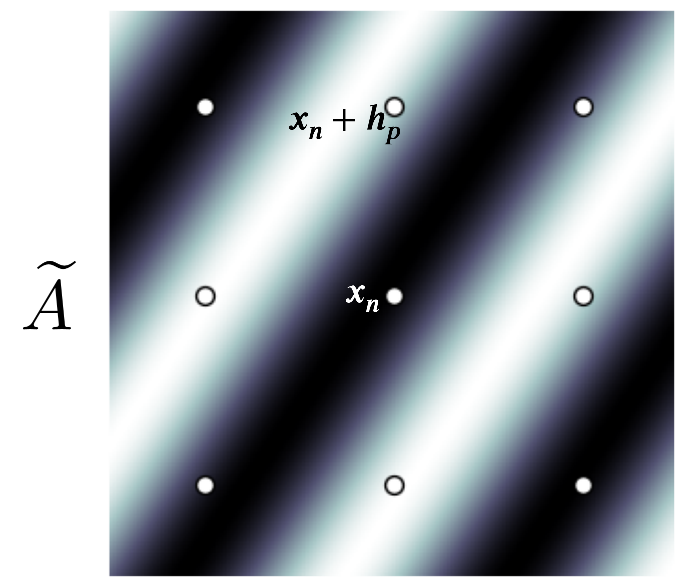


$$\big(F \star \mathrm{Re}(\Psi)\big)(\boldsymbol{x_n} + \boldsymbol{h_p}) \approx \underbrace{|(F \star \Psi)(\boldsymbol{x_n})|}_{Y^{\mathrm{mod}}[\boldsymbol{n}]} \cos\big(\langle \boldsymbol{\nu}, \boldsymbol{h_p} \rangle - H(\boldsymbol{x_n})\big)$$

Adaptation to the discrete case

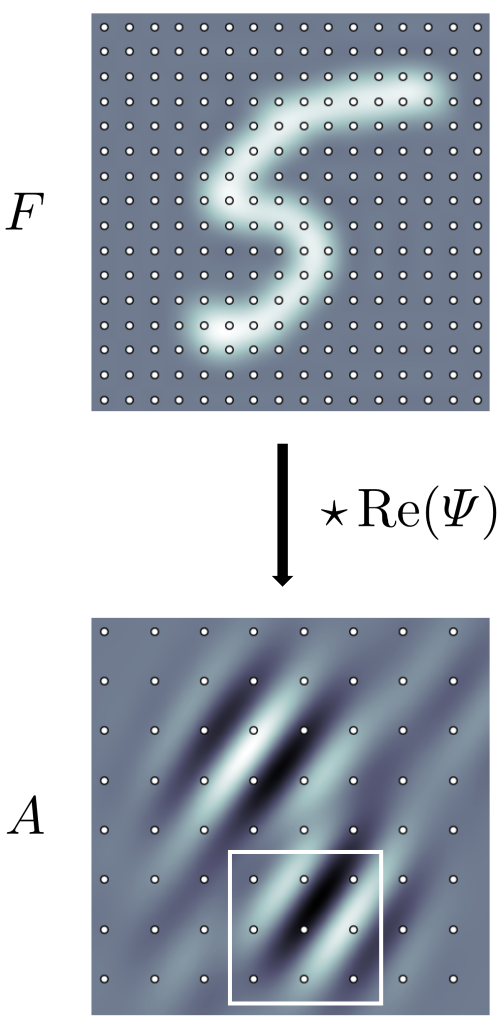


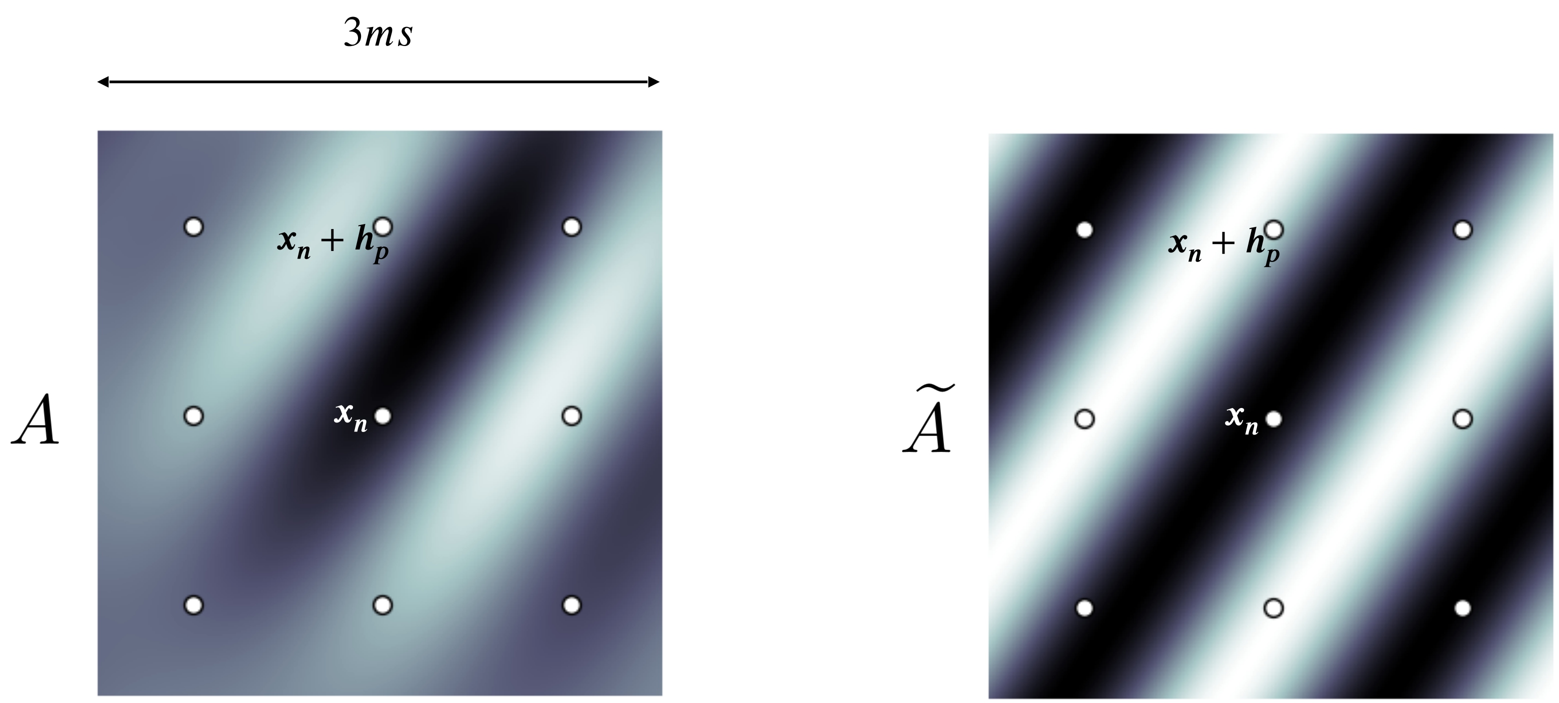




$$\underbrace{\big(F \star \mathrm{Re}(\Psi)\big)(\boldsymbol{x_n} + \boldsymbol{h_p})}_{\max\limits_{\|\boldsymbol{p}\| \leq q}} \approx \underbrace{|(F \star \Psi)(\boldsymbol{x_n})|}_{\Upsilon^{\mathrm{mod}}[\boldsymbol{n}]} \underbrace{\cos\big(\langle \boldsymbol{\nu}, \boldsymbol{h_p} \rangle - H(\boldsymbol{x_n})\big)}_{\max\limits_{\|\boldsymbol{p}\| \leq q}}$$

Adaptation to the discrete case

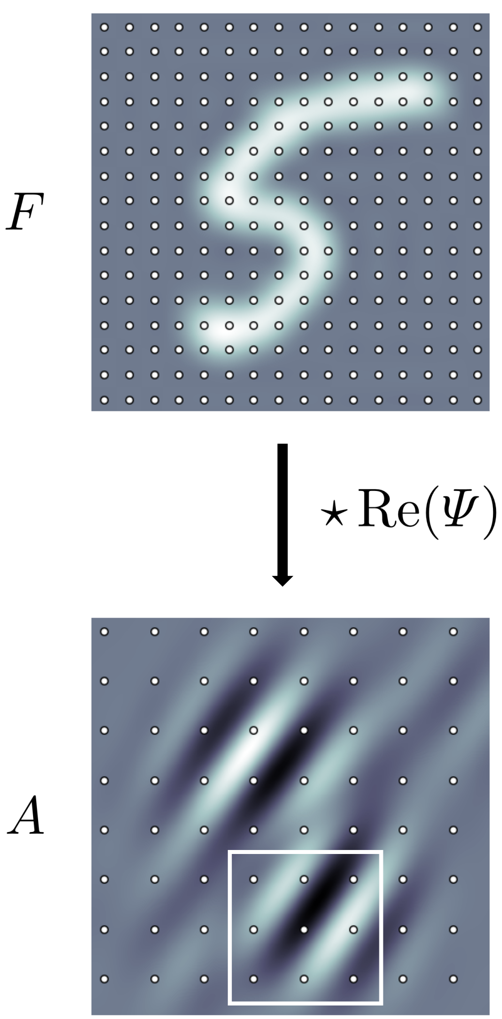


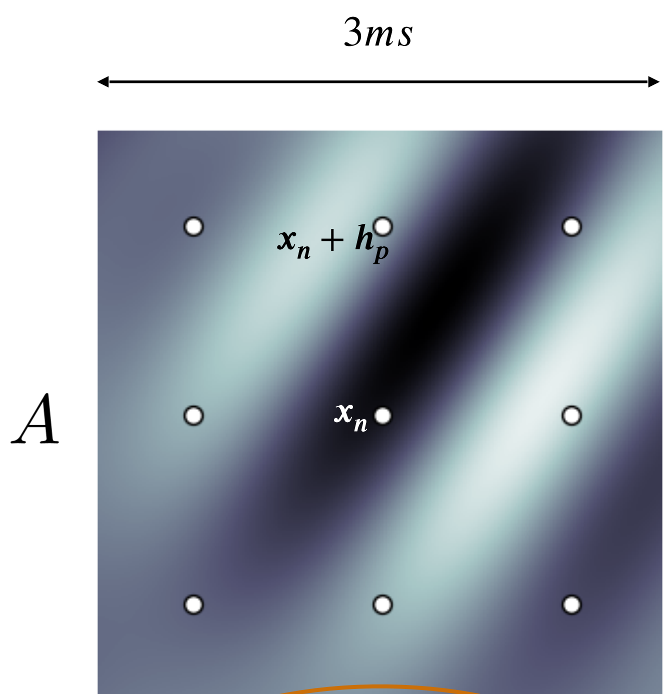


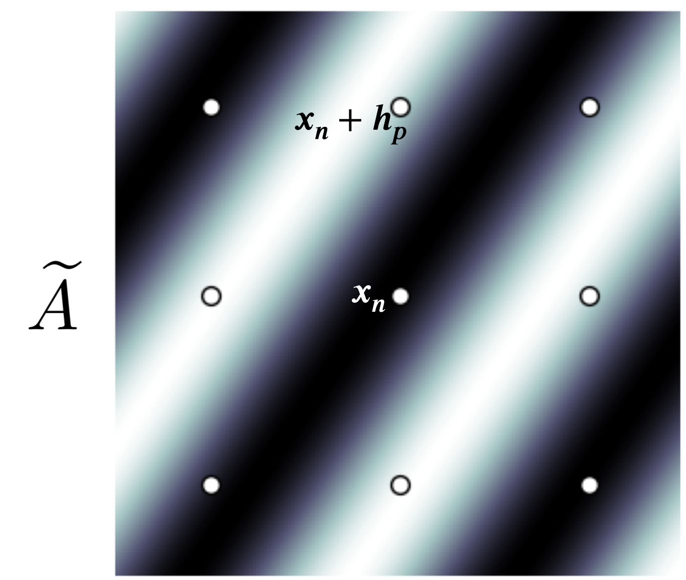
$$\underbrace{\big(F \star \mathrm{Re}(\Psi)\big)(\boldsymbol{x_n} + \boldsymbol{h_p})}_{\max\limits_{\|\boldsymbol{p}\| \le q}} \approx \underbrace{\big|(F \star \Psi)(\boldsymbol{x_n})\big|}_{\mathrm{Y}^{\mathrm{mod}}[\boldsymbol{n}]} \underbrace{\cos\big(\langle \boldsymbol{\nu}, \boldsymbol{h_p}\rangle - H(\boldsymbol{x_n})\big)}_{\max\limits_{\|\boldsymbol{p}\| \le q}}$$

Maximum value
over a discrete grid
($q = 1$ in general)

Adaptation to the discrete case







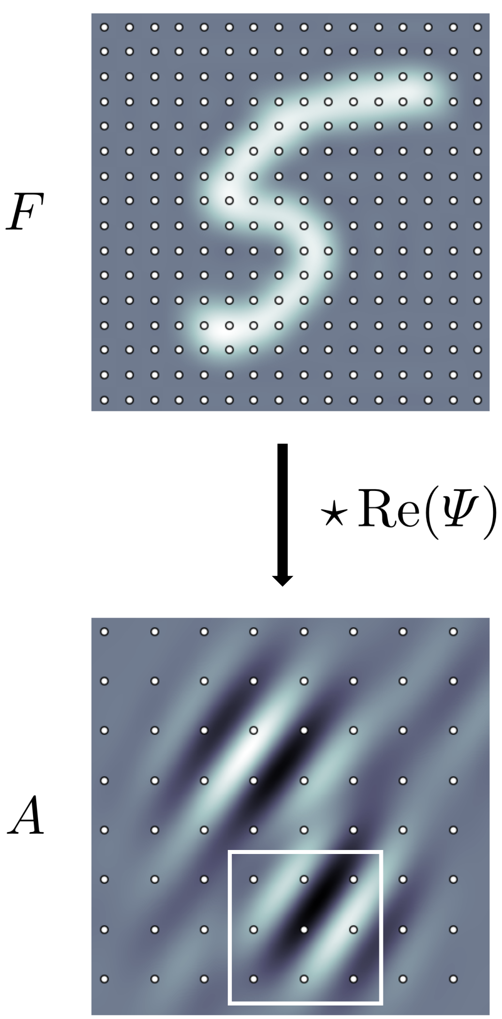
$$\underbrace{\big(F \star \mathrm{Re}(\Psi)\big)(\boldsymbol{x_n} + \boldsymbol{h_p})}_{\max\limits_{\|\boldsymbol{p}\| \leq q}} \approx \big|(F \star \Psi)(\boldsymbol{x_n})\big| \underbrace{\cos\big(\langle \boldsymbol{\nu}, \boldsymbol{h_p}\rangle - H(\boldsymbol{x_n})\big)}_{\max\limits_{\|\boldsymbol{p}\| \leq q}}$$

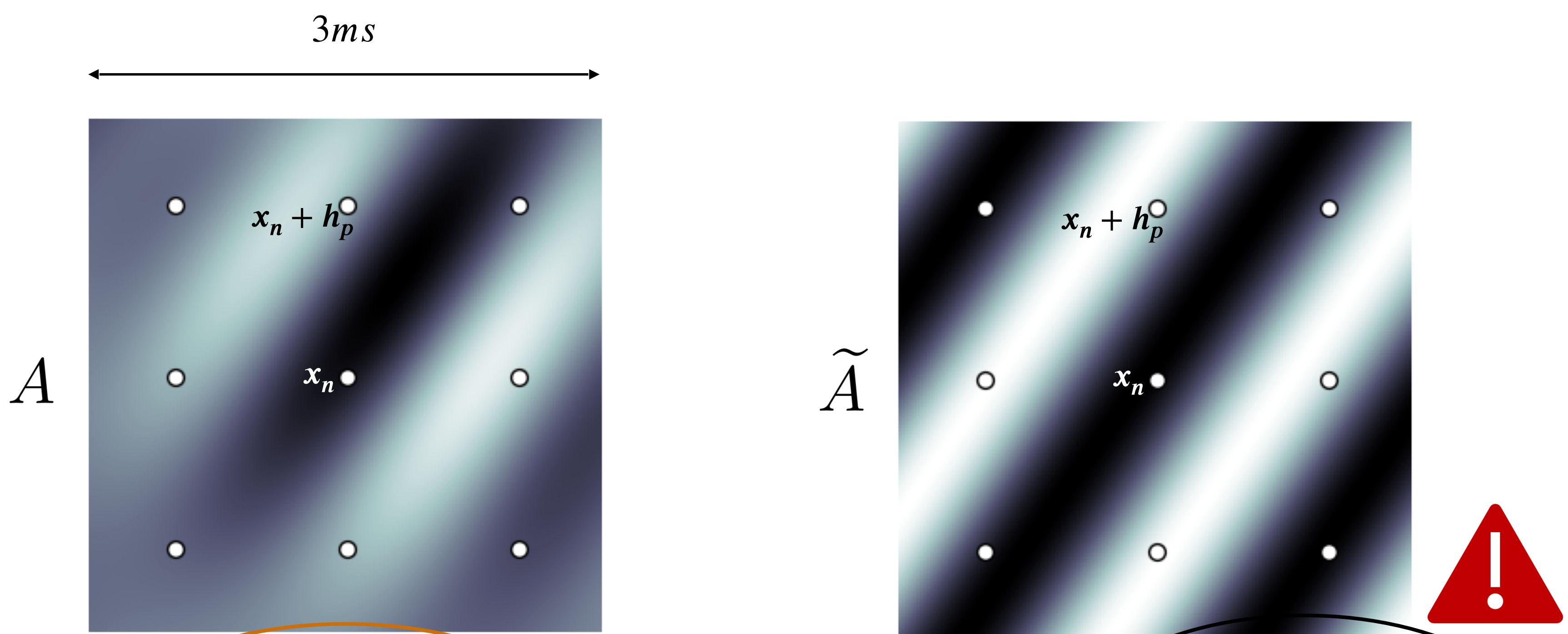
$\mathrm{Y}^{\max}[\boldsymbol{n}]$

$\mathrm{Y}^{\mathrm{mod}}[\boldsymbol{n}]$

Maximum value
over a discrete grid
($q = 1$ in general)

Adaptation to the discrete case





$$\underbrace{\big(F \star \mathrm{Re}(\Psi)\big)(\boldsymbol{x_n} + \boldsymbol{h_p})}_{\max\limits_{\|\boldsymbol{p}\| \le q}} \approx \big|(F \star \Psi)(\boldsymbol{x_n})\big| \underbrace{\cos\big(\langle \boldsymbol{\nu}, \boldsymbol{h_p}\rangle - H(\boldsymbol{x_n})\big)}_{\max\limits_{\|\boldsymbol{p}\| \le q}}$$

$\mathrm{Y}^{\max}[\boldsymbol{n}]$ $\mathrm{Y}^{\mathrm{mod}}[\boldsymbol{n}]$ $G^{\max}(\boldsymbol{x_n}) \le 1$

Maximum value over a discrete grid ($q = 1$ in general)

Adaptation to the discrete case

$$q \ll 2\pi/(m\kappa) \quad \Longrightarrow \quad U_{m,\,q}^{\max}\mathrm{X}[\boldsymbol{n}] \approx U_{2m}^{\mathrm{mod}}\mathrm{X}[\boldsymbol{n}] \max_{\|\boldsymbol{p}\|_\infty \leq q} G_{\mathrm{X}}\big(\boldsymbol{x_n},\, \boldsymbol{h_p}\big),$$

Adaptation to the discrete case

$$q \ll 2\pi/(m\kappa) \quad \Longrightarrow \quad U_{m,\,q}^{\max}\mathrm{X}[\boldsymbol{n}] \approx U_{2m}^{\mathrm{mod}}\mathrm{X}[\boldsymbol{n}] \max_{\|\boldsymbol{p}\|_\infty \leq q} G_{\mathrm{X}}\left(\boldsymbol{x_n},\, \boldsymbol{h_p}\right),$$

Not necessarily reaches 1

Adaptation to the discrete case

$$q \ll 2\pi/(m\kappa) \quad \Longrightarrow \quad U^{\max}_{m,\,q}\mathrm{X}[\boldsymbol{n}] \approx U^{\mathrm{mod}}_{2m}\mathrm{X}[\boldsymbol{n}] \max_{\|\boldsymbol{p}\|_\infty \leq q} G_{\mathrm{X}}\big(\boldsymbol{x_n},\, \boldsymbol{h_p}\big),$$

Not necessarily reaches 1

$$q \ll 2\pi/(m\kappa) \Longrightarrow \big\|U^{\mathrm{mod}}_{2m}\mathrm{X} - U^{\max}_{m,\,q}\mathrm{X}\big\|_2 \approx \|\delta_{m,\,q}\mathrm{X}\|_2$$

Adaptation to the discrete case

$$q \ll 2\pi/(m\kappa) \quad \Longrightarrow \quad U^{\max}_{m,\,q}\mathrm{X}[\boldsymbol{n}] \approx U^{\mathrm{mod}}_{2m}\mathrm{X}[\boldsymbol{n}] \max_{\|\boldsymbol{p}\|_\infty \leq q} G_{\mathrm{X}}\big(\boldsymbol{x_n},\, \boldsymbol{h_p}\big),$$

Not necessarily reaches 1

$$q \ll 2\pi/(m\kappa) \implies \left\|U^{\mathrm{mod}}_{2m}\mathrm{X} - U^{\max}_{m,\,q}\mathrm{X}\right\|_2 \approx \|\delta_{m,\,q}\mathrm{X}\|_2$$

$$\delta_{m,\,q}\mathrm{X}[\boldsymbol{n}] := U^{\mathrm{mod}}_{2m}\mathrm{X}[\boldsymbol{n}] \left(1 - \max_{\|\boldsymbol{p}\|_\infty \leq q} G_{\mathrm{X}}\big(\boldsymbol{x_n},\, \boldsymbol{h_p}\big)\right)$$

Adaptation to the discrete case

$$q \ll 2\pi/(m\kappa) \quad \Longrightarrow \quad U^{\max}_{m,\,q}\mathrm{X}[\boldsymbol{n}] \approx U^{\mathrm{mod}}_{2m}\mathrm{X}[\boldsymbol{n}] \max_{\|\boldsymbol{p}\|_\infty \leq q} G_{\mathrm{X}}\big(\boldsymbol{x}_{\boldsymbol{n}},\, \boldsymbol{h}_{\boldsymbol{p}}\big),$$

Not necessarily reaches 1

$$q \ll 2\pi/(m\kappa) \implies \left\|U^{\mathrm{mod}}_{2m}\mathrm{X} - U^{\max}_{m,\,q}\mathrm{X}\right\|_2 \approx \|\delta_{m,\,q}\mathrm{X}\|_2$$

$$\delta_{m,\,q}\mathrm{X}[\boldsymbol{n}] := U^{\mathrm{mod}}_{2m}\mathrm{X}[\boldsymbol{n}] \left(1 - \max_{\|\boldsymbol{p}\|_\infty \leq q} G_{\mathrm{X}}\big(\boldsymbol{x}_{\boldsymbol{n}},\, \boldsymbol{h}_{\boldsymbol{p}}\big)\right)$$

Theorem (Bound on the difference of $\mathbb{C}$**Mod** and $\mathbb{R}$**Max**)

If $\kappa \leq \pi/m$ and under another reasonable hypothesis

$$\left\|U^{\mathrm{mod}}_{2m}\mathrm{X} - U^{\max}_{m,\,q}\mathrm{X}\right\|_2 \leq \left\|\delta_{m,\,q}\mathrm{X}\right\|_2 + \beta_q(m\kappa)\left\|U^{\mathrm{mod}}_{2m}\mathrm{X}\right\|_2$$

Adaptation to the discrete case

$$q \ll 2\pi/(m\kappa) \quad \Longrightarrow \quad U^{\max}_{m,\,q}\mathrm{X}[\boldsymbol{n}] \approx U^{\mathrm{mod}}_{2m}\mathrm{X}[\boldsymbol{n}] \max_{\|\boldsymbol{p}\|_\infty \leq q} G_{\mathrm{X}}\big(\boldsymbol{x}_{\boldsymbol{n}},\, \boldsymbol{h}_{\boldsymbol{p}}\big),$$

Not necessarily reaches 1

$$q \ll 2\pi/(m\kappa) \Longrightarrow \left\|U^{\mathrm{mod}}_{2m}\mathrm{X} - U^{\max}_{m,\,q}\mathrm{X}\right\|_2 \approx \left\|\delta_{m,\,q}\mathrm{X}\right\|_2$$

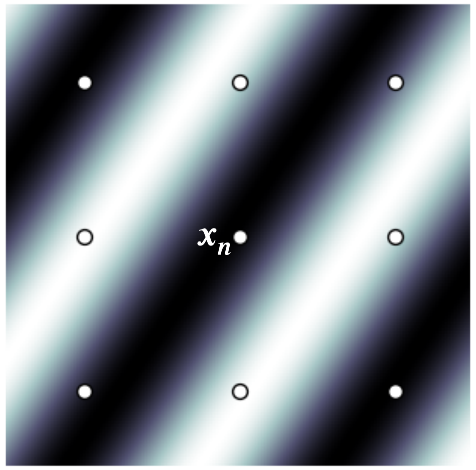
$$\delta_{m,\,q}\mathrm{X}[\boldsymbol{n}] := U^{\mathrm{mod}}_{2m}\mathrm{X}[\boldsymbol{n}]\left(1 - \max_{\|\boldsymbol{p}\|_\infty \leq q} G_{\mathrm{X}}\big(\boldsymbol{x}_{\boldsymbol{n}},\, \boldsymbol{h}_{\boldsymbol{p}}\big)\right) \qquad \beta_q : \kappa' \mapsto q\kappa'.$$

Theorem (Bound on the difference of $\mathbb{C}$**Mod** and $\mathbb{R}$**Max**)

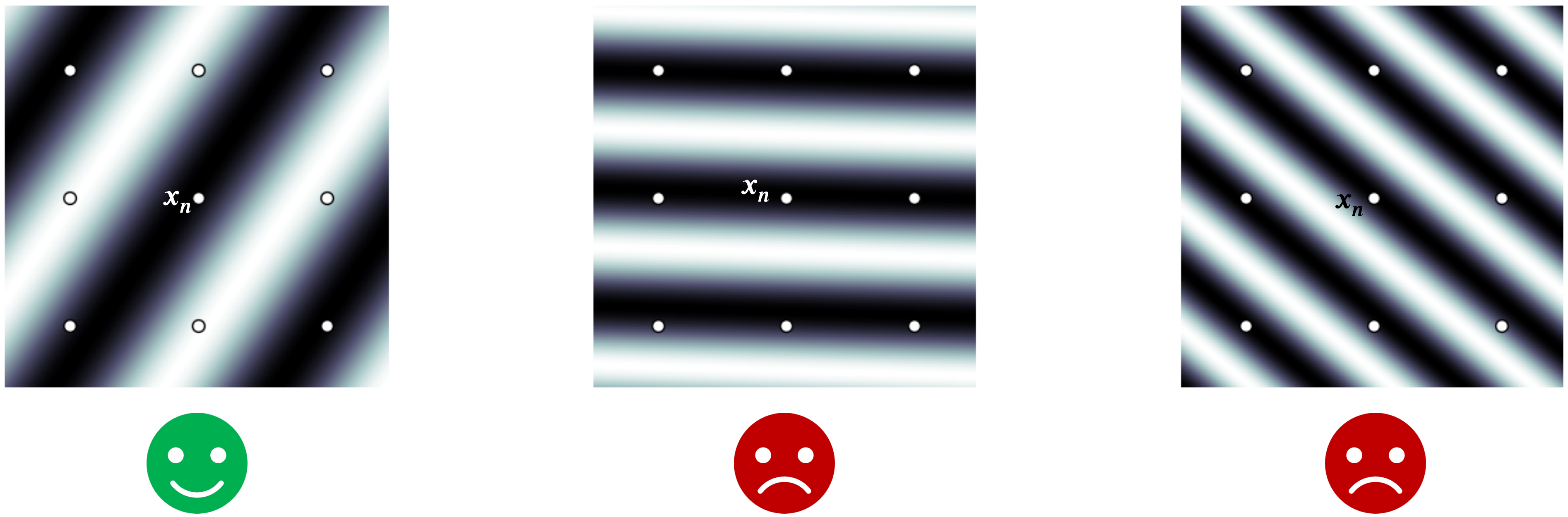
If $\kappa \leq \pi/m$ and under another reasonable hypothesis

$$\left\|U^{\mathrm{mod}}_{2m}\mathrm{X} - U^{\max}_{m,\,q}\mathrm{X}\right\|_2 \leq \left\|\delta_{m,\,q}\mathrm{X}\right\|_2 + \beta_q(m\kappa)\left\|U^{\mathrm{mod}}_{2m}\mathrm{X}\right\|_2$$

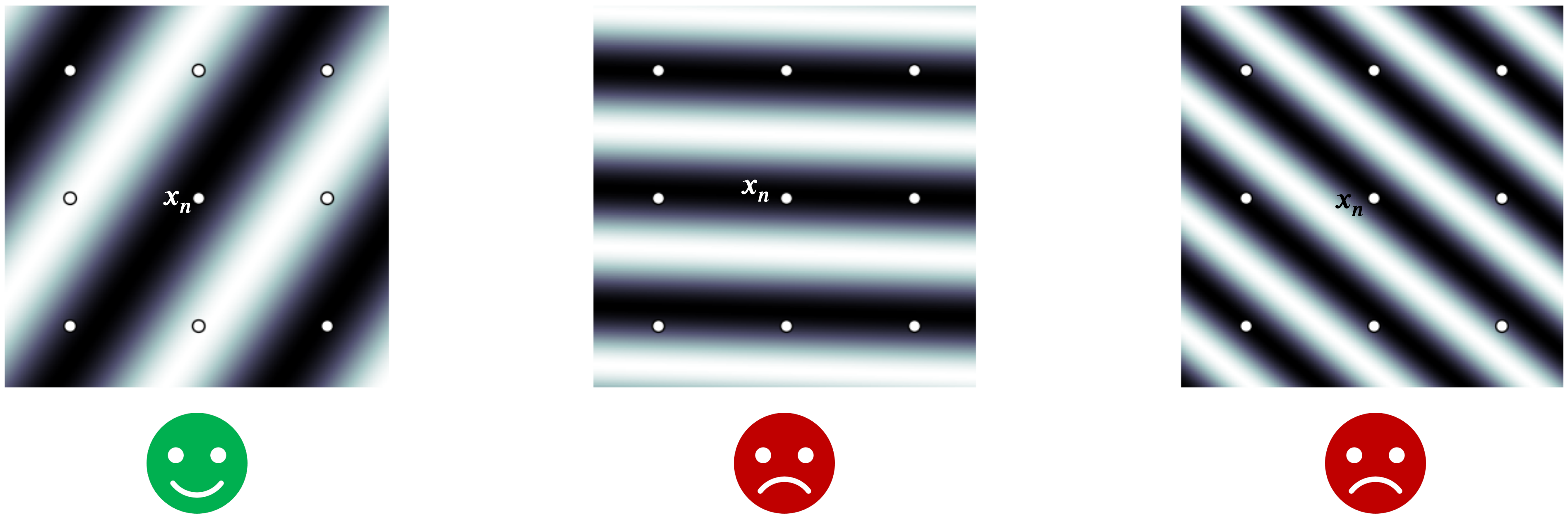
Pathological frequencies



Pathological frequencies

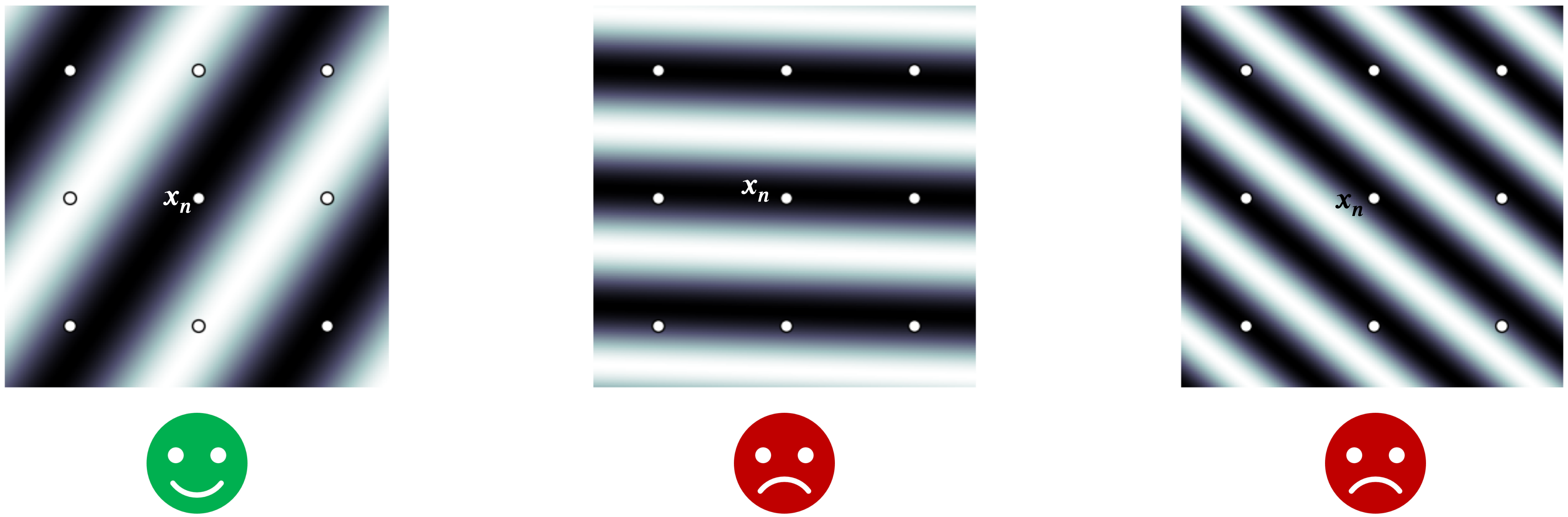


Pathological frequencies



- Need for a probabilistic framework where X (resp. F) is seen as a discrete (resp. continuous) stochastic process on $\mathbb{Z}^2$ (resp. $\mathbb{R}^2$)

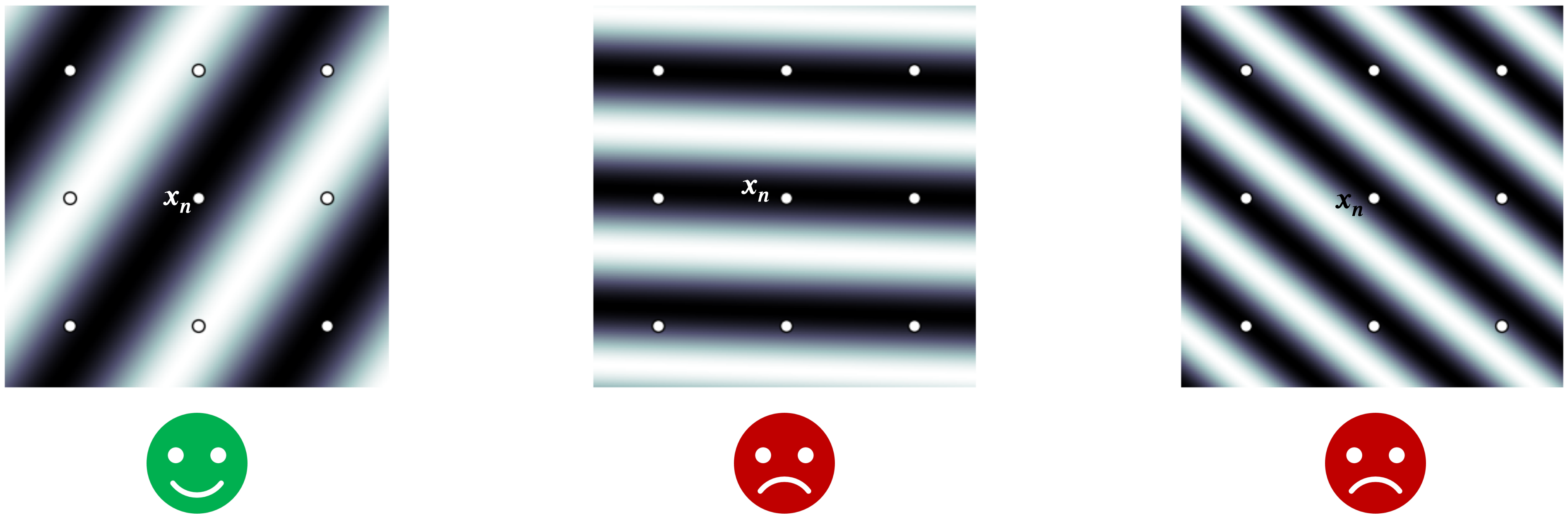
Pathological frequencies



- Need for a probabilistic framework where X (resp. F) is seen as a discrete (resp. continuous) stochastic process on $\mathbb{Z}^2$ (resp. $\mathbb{R}^2$)

- Quantity of interest: $\mathbb{E}\left[\left(1 - G^{\max}(\boldsymbol{x_n})\right)^2\right]$

Pathological frequencies

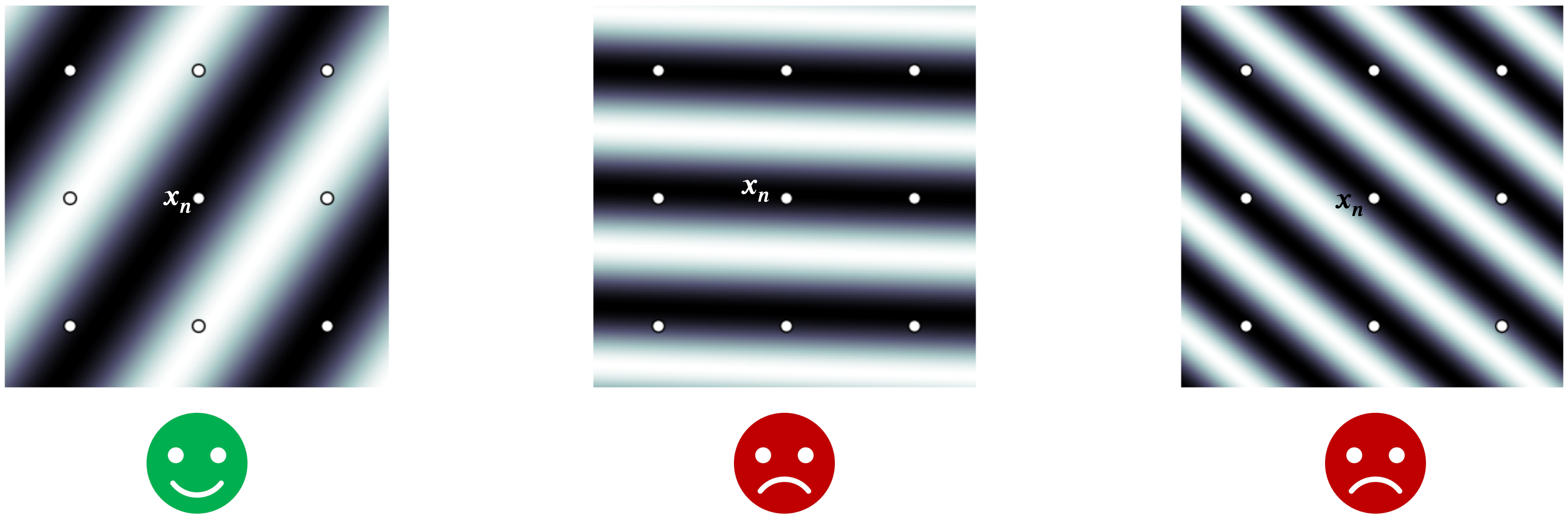


- Need for a probabilistic framework where X (resp. F) is seen as a discrete (resp. continuous) stochastic process on $\mathbb{Z}^2$ (resp. $\mathbb{R}^2$)

- Quantity of interest: $\mathbb{E}\left[\left(1 - G^{\max}(\boldsymbol{x_n})\right)^2\right]$

with $\quad G^{\max}(\boldsymbol{x_n}) = \max\limits_{\|\boldsymbol{p}\|_\infty \leq 1} \cos\left(\langle \boldsymbol{\nu}, \boldsymbol{h_p} \rangle - H(\boldsymbol{x_n})\right)$

Pathological frequencies



- Need for a probabilistic framework where X (resp. F) is seen as a discrete (resp. continuous) stochastic process on $\mathbb{Z}^2$ (resp. $\mathbb{R}^2$)

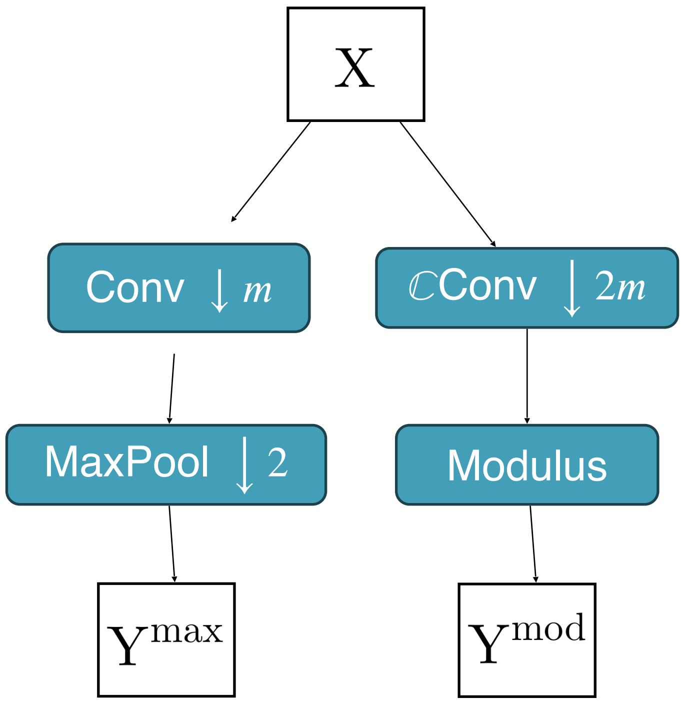
- Quantity of interest: $\mathbb{E}\left[\left(1 - G^{\max}(\boldsymbol{x_n})\right)^2\right]$

Hypothesis: uniformly distributed

with $G^{\max}(\boldsymbol{x_n}) = \max_{\|\boldsymbol{p}\|_\infty \leq 1} \cos\left(\langle \boldsymbol{\nu}, \boldsymbol{h_p} \rangle - H(\boldsymbol{x_n})\right)$

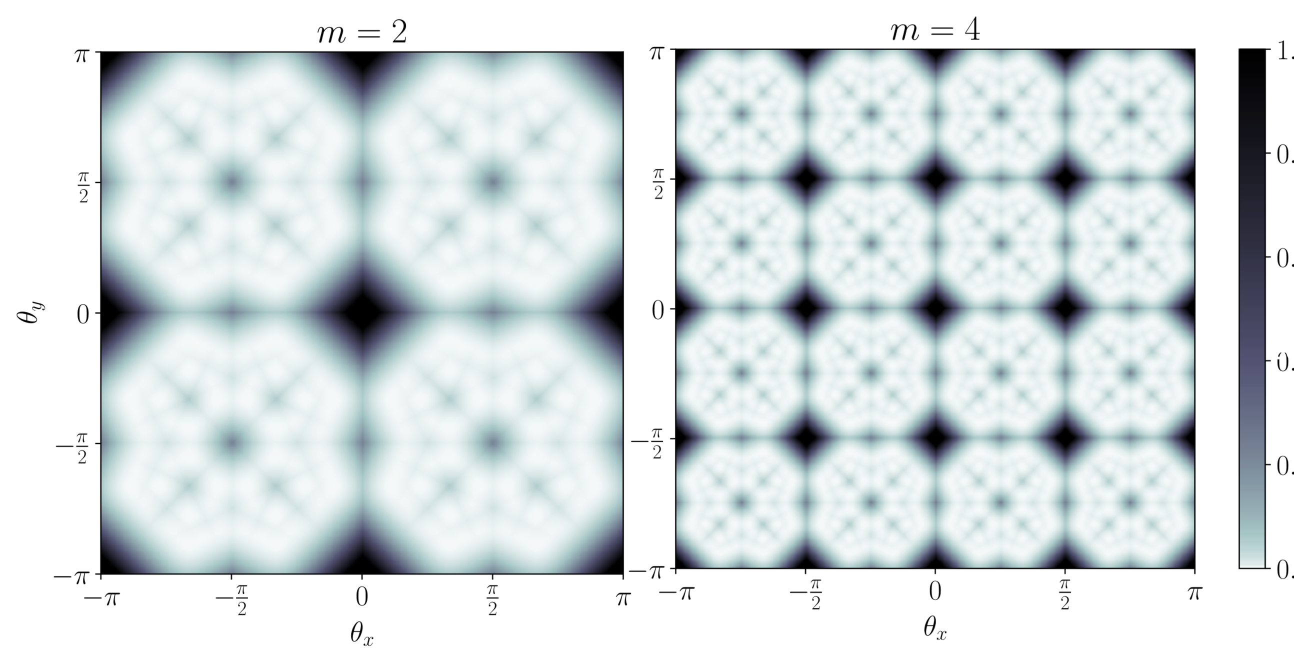
Main result

- MSE between $\mathbb{C}$**Mod** and $\mathbb{R}$**Max** output



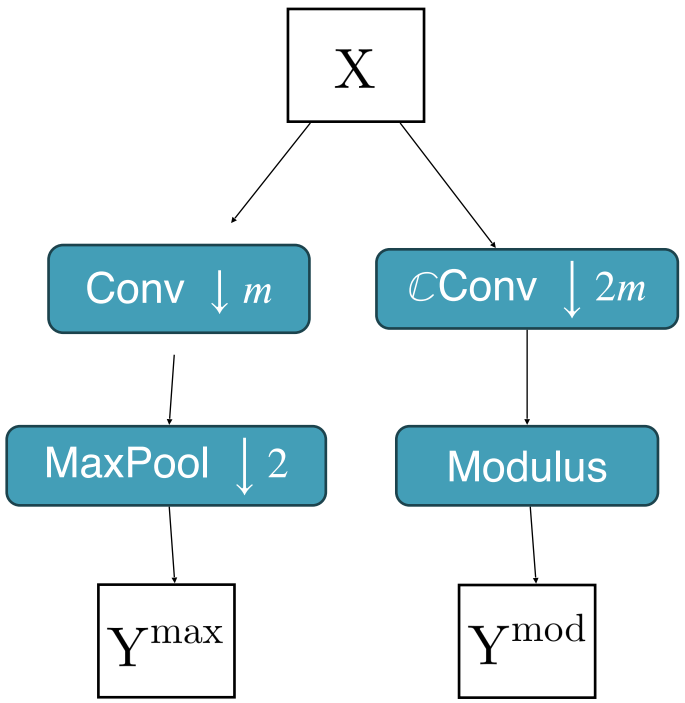
$$\mathbb{E}\left[\frac{\left\|Y^{\max} - Y^{\text{mod}}\right\|_2^2}{\left\|Y^{\text{mod}}\right\|_2^2}\right] \leq \left(\beta_q(m\kappa) + \gamma_q(m\boldsymbol{\theta})\right)^2$$

$$\boldsymbol{\theta} \mapsto \gamma_q(m\boldsymbol{\theta})^2$$
$$q = 1$$



Main result

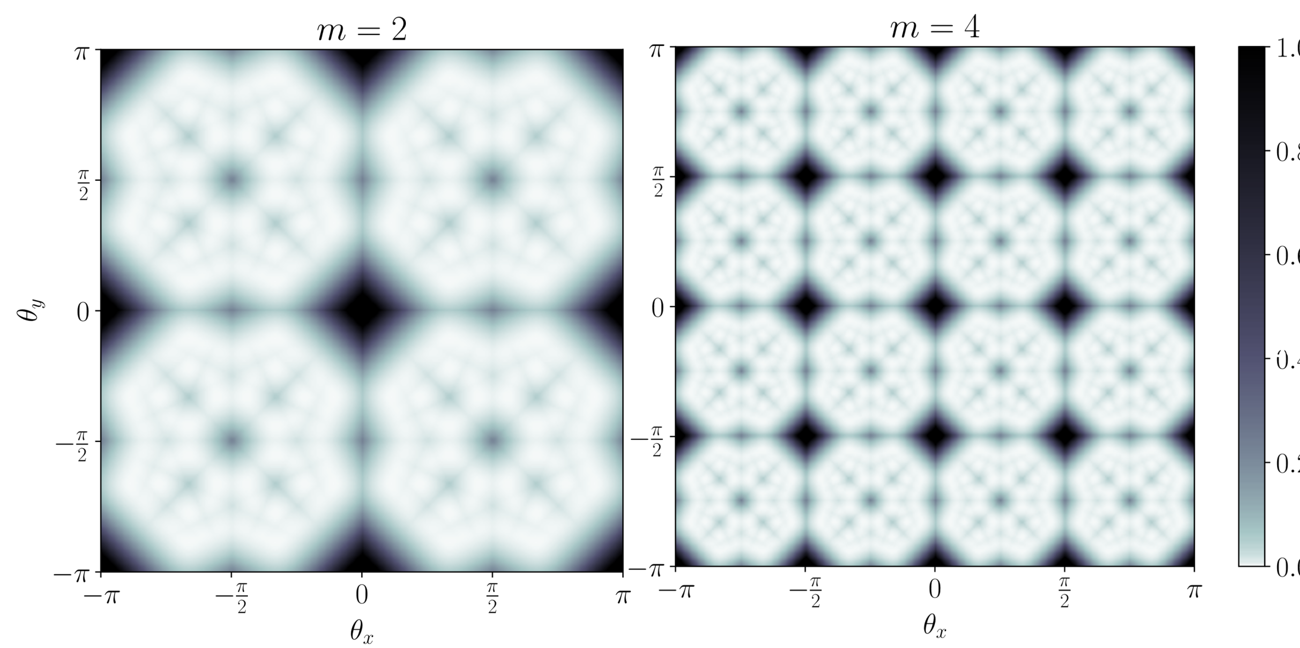
- MSE between $\mathbb{C}$**Mod** and $\mathbb{R}$**Max** output



$$\mathbb{E}\left[\frac{\left\|Y^{\max} - Y^{\mathrm{mod}}\right\|_2^2}{\left\|Y^{\mathrm{mod}}\right\|_2^2}\right] \leq \left(\beta_q(m\kappa) + \gamma_q(m\boldsymbol{\theta})\right)^2$$

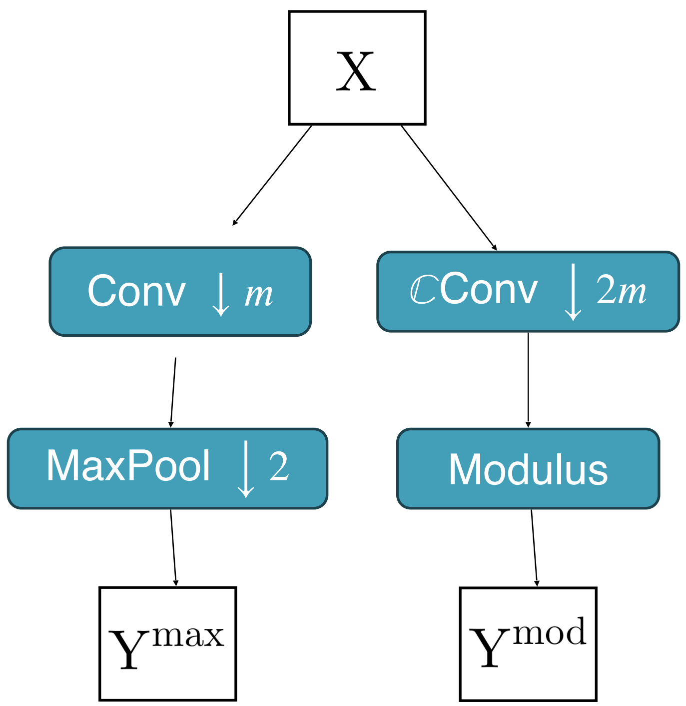
Upper bound under hypothesis $G^{\max} = 1$

$$\boldsymbol{\theta} \mapsto \gamma_q(m\boldsymbol{\theta})^2$$
$$q = 1$$



Main result

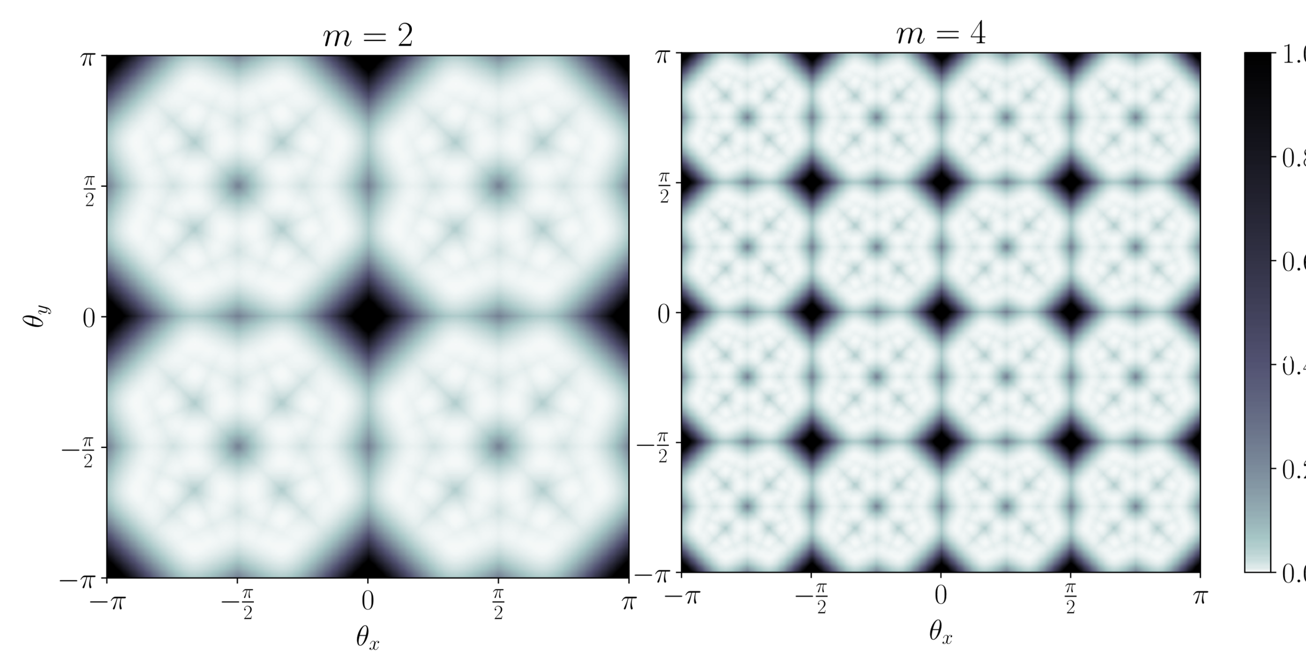
- MSE between $\mathbb{C}$**Mod** and $\mathbb{R}$**Max** output



$$\mathbb{E}\left[\frac{\left\|\mathrm{Y}^{\max} - \mathrm{Y}^{\mathrm{mod}}\right\|_2^2}{\left\|\mathrm{Y}^{\mathrm{mod}}\right\|_2^2}\right] \leq \left(\beta_q(m\kappa) + \gamma_q(m\boldsymbol{\theta})\right)^2$$

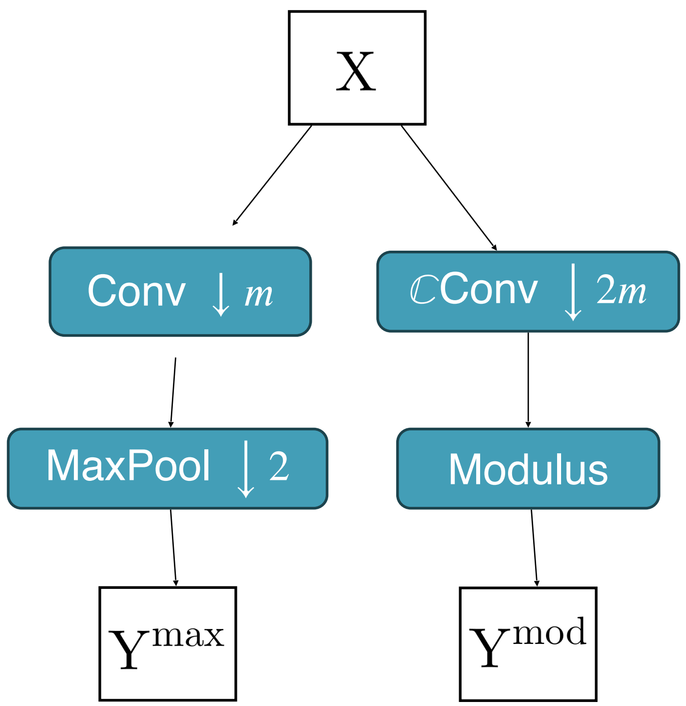
Discrete nature of the max pooling grid

$$\mathbb{E}\left[\left(1 - G^{\max}(\boldsymbol{x_n})\right)^2\right] \quad \boldsymbol{\theta} \mapsto \gamma_q(m\boldsymbol{\theta})^2, \quad q = 1$$



Main result

- MSE between $\mathbb{C}$**Mod** and $\mathbb{R}$**Max** output

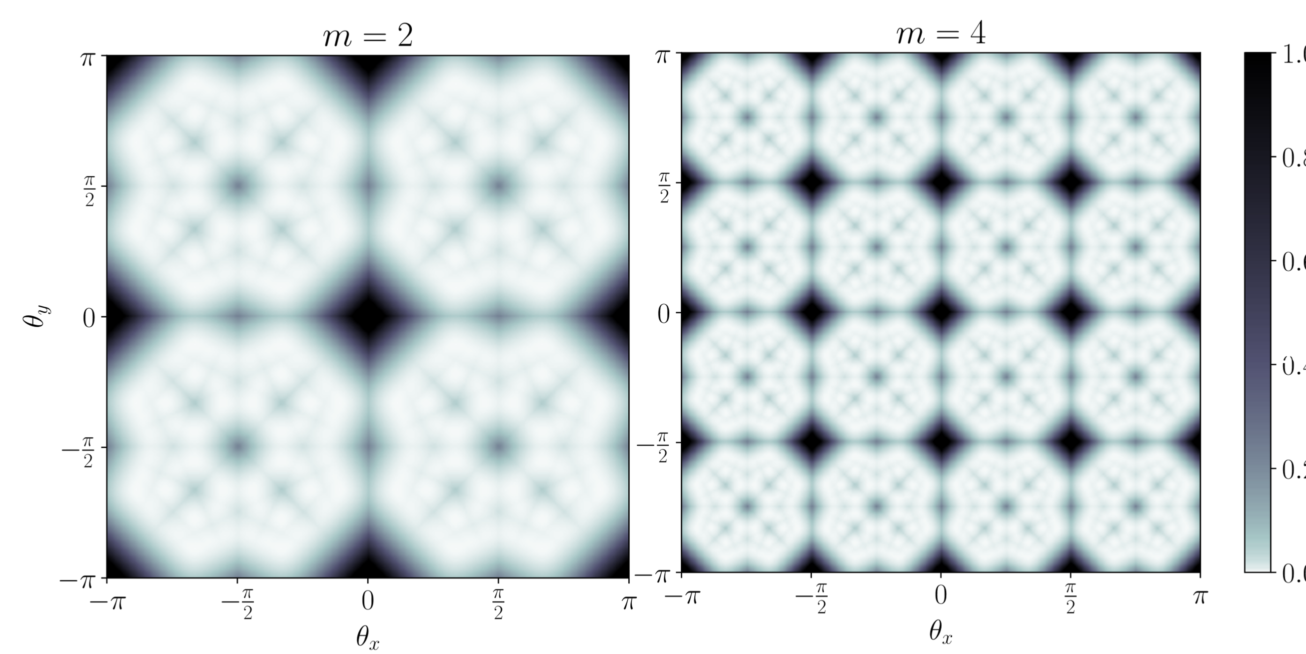


$$\mathbb{E}\left[\frac{\left\|\mathrm{Y}^{\max}-\mathrm{Y}^{\mathrm{mod}}\right\|_2^2}{\left\|\mathrm{Y}^{\mathrm{mod}}\right\|_2^2}\right] \leq \left(\beta_q(m\kappa)+\gamma_q(m\boldsymbol{\theta})\right)^2$$

Discrete nature of the max pooling grid

$$\mathbb{E}\left[\left(1-G^{\max}(\boldsymbol{x_n})\right)^2\right] \rightarrow \quad \boldsymbol{\theta} \mapsto \gamma_q(m\boldsymbol{\theta})^2, \quad q=1$$

$$\gamma_q(\boldsymbol{\omega}) = \sqrt{\frac{3}{2}+\frac{1}{4\pi}\sum_{i=0}^{n_q-1}\left(\sin\delta H_i^{(q)}(\boldsymbol{\omega})-8\sin\frac{\delta H_i^{(q)}(\boldsymbol{\omega})}{2}\right)}$$



Sketch of the proof

- **Reformulation** of the problem on the unit circle $\mathbb{S}^1 \subset \mathbb{C}$

Sketch of the proof

- **Reformulation** of the problem on the unit circle $\mathbb{S}^1 \subset \mathbb{C}$

  $[z,\, z']_{\mathbb{S}^1} \subset \mathbb{S}^1$ arc on the unit circle going from z to z'

Sketch of the proof

- **Reformulation** of the problem on the unit circle $\mathbb{S}^1 \subset \mathbb{C}$

$[z,\, z']_{\mathbb{S}^1} \subset \mathbb{S}^1$ arc on the unit circle going from z to z'

$$Z_{\mathrm{X}} : \boldsymbol{x} \mapsto \mathrm{e}^{i\, H_{\mathrm{X}}(\boldsymbol{x})} \qquad \text{and} \qquad Z_{\boldsymbol{p}} : \boldsymbol{\omega} \mapsto= \mathrm{e}^{i\langle \boldsymbol{\omega},\, \boldsymbol{p}\rangle}$$

Sketch of the proof

- **Reformulation** of the problem on the unit circle $\mathbb{S}^1 \subset \mathbb{C}$

  $[z,\, z']_{\mathbb{S}^1} \subset \mathbb{S}^1$ arc on the unit circle going from z to z'

  $$Z_{\mathrm{X}} : \boldsymbol{x} \mapsto \mathrm{e}^{i\, H_{\mathrm{X}}(\boldsymbol{x})} \qquad \text{and} \qquad Z_{\boldsymbol{p}} : \boldsymbol{\omega} \mapsto = \mathrm{e}^{i\langle \boldsymbol{\omega},\, \boldsymbol{p}\rangle}$$

- $G_{\mathrm{X}}\big(\boldsymbol{x_n},\, \boldsymbol{h_p}\big) = \mathrm{Re}\big(Z_{\mathrm{X}}^*(\boldsymbol{x_n})\, Z_{\boldsymbol{p}}(m\boldsymbol{\theta})\big) \longrightarrow \cos\big(\langle \boldsymbol{\nu},\, \boldsymbol{h_p}\rangle - H(\boldsymbol{x_n})\big)$

Sketch of the proof

- **Reformulation** of the problem on the unit circle $\mathbb{S}^1 \subset \mathbb{C}$

  $[z,\, z']_{\mathbb{S}^1} \subset \mathbb{S}^1$ arc on the unit circle going from z to z'

  $\boldsymbol{h_p} = ms\boldsymbol{p}$

  $Z_{\mathrm{X}} : \boldsymbol{x} \mapsto \mathrm{e}^{i\, H_{\mathrm{X}}(\boldsymbol{x})}$ and $Z_{\boldsymbol{p}} : \boldsymbol{\omega} \mapsto = \mathrm{e}^{i\langle \boldsymbol{\omega},\, \boldsymbol{p}\rangle}$

- $G_{\mathrm{X}}\big(\boldsymbol{x_n},\, \boldsymbol{h_p}\big) = \mathrm{Re}\big(Z_{\mathrm{X}}^{*}(\boldsymbol{x_n})\, Z_{\boldsymbol{p}}(m\boldsymbol{\theta})\big)$ ⟶ $\cos\big(\langle \boldsymbol{\nu},\, \boldsymbol{h_p}\rangle - H(\boldsymbol{x_n})\big)$

Sketch of the proof

- **Reformulation** of the problem on the unit circle $\mathbb{S}^1 \subset \mathbb{C}$

$[z,\, z']_{\mathbb{S}^1} \subset \mathbb{S}^1$ arc on the unit circle going from z to z'

$\boldsymbol{h_p} = ms\boldsymbol{p}$

$Z_{\mathrm{X}} : \boldsymbol{x} \mapsto \mathrm{e}^{i\, H_{\mathrm{X}}(\boldsymbol{x})}$ and $Z_{\boldsymbol{p}} : \boldsymbol{\omega} \mapsto= \mathrm{e}^{i\langle \boldsymbol{\omega},\, \boldsymbol{p}\rangle}$

$\boldsymbol{\nu} = \boldsymbol{\theta}/s$

- $G_{\mathrm{X}}\big(\boldsymbol{x_n},\, \boldsymbol{h_p}\big) = \mathrm{Re}\big(Z^*_{\mathrm{X}}(\boldsymbol{x_n})\, Z_{\boldsymbol{p}}(m\boldsymbol{\theta})\big)$ ⟶ $\cos\big(\langle \boldsymbol{\nu},\, \boldsymbol{h_p}\rangle - H(\boldsymbol{x_n})\big)$

Sketch of the proof

- **Reformulation** of the problem on the unit circle $\mathbb{S}^1 \subset \mathbb{C}$

  $[z,\, z']_{\mathbb{S}^1} \subset \mathbb{S}^1$ arc on the unit circle going from z to z'

  $\boldsymbol{h_p} = ms\boldsymbol{p}$

  $Z_{\mathrm{X}} : \boldsymbol{x} \mapsto \mathrm{e}^{i\, H_{\mathrm{X}}(\boldsymbol{x})}$ and $Z_{\boldsymbol{p}} : \boldsymbol{\omega} \mapsto = \mathrm{e}^{i\langle \boldsymbol{\omega},\, \boldsymbol{p}\rangle}$

  $\boldsymbol{\nu} = \boldsymbol{\theta}/s$

- $G_{\mathrm{X}}\big(\boldsymbol{x_n},\, \boldsymbol{h_p}\big) = \mathrm{Re}\big(Z^*_{\mathrm{X}}(\boldsymbol{x_n})\, Z_{\boldsymbol{p}}(m\boldsymbol{\theta})\big)$ ⟶ $\cos\big(\langle \boldsymbol{\nu},\, \boldsymbol{h_p}\rangle - H(\boldsymbol{x_n})\big)$

- Sort $\{Z_{\boldsymbol{p}}(\boldsymbol{\omega})\}_{\boldsymbol{p}\in\{-q..q\}^2}$ ⟶ $\big(Z_i^{(q)}(\boldsymbol{\omega})\big)_{i\in\{0..n_q-1\}}$ $\quad n_q := (2q+1)^2$

Sketch of the proof

- **Reformulation** of the problem on the unit circle $\mathbb{S}^1 \subset \mathbb{C}$

  $[z,\, z']_{\mathbb{S}^1} \subset \mathbb{S}^1$ arc on the unit circle going from z to z'

  $$\boldsymbol{h_p} = ms\boldsymbol{p}$$

  $$Z_{\mathrm{X}} : \boldsymbol{x} \mapsto \mathrm{e}^{i\, H_{\mathrm{X}}(\boldsymbol{x})} \qquad \text{and} \qquad Z_{\boldsymbol{p}} : \boldsymbol{\omega} \mapsto = \mathrm{e}^{i\langle \boldsymbol{\omega},\, \boldsymbol{p}\rangle}$$

  $$\boldsymbol{\nu} = \boldsymbol{\theta}/s$$

- $G_{\mathrm{X}}\big(\boldsymbol{x_n},\, \boldsymbol{h_p}\big) = \mathrm{Re}\big(Z^*_{\mathrm{X}}(\boldsymbol{x_n})\, Z_{\boldsymbol{p}}(m\boldsymbol{\theta})\big)$ ⟶ $\cos\big(\langle \boldsymbol{\nu},\, \boldsymbol{h_p}\rangle - H(\boldsymbol{x_n})\big)$

- Sort $\{Z_{\boldsymbol{p}}(\boldsymbol{\omega})\}_{\boldsymbol{p}\in\{-q..q\}^2}$ ⟶ $\big(Z_i^{(q)}(\boldsymbol{\omega})\big)_{i\in\{0..n_q-1\}}$ $\qquad n_q := (2q+1)^2$

  in ascending order of their argument:

  $$0 = H_0^{(q)}(\boldsymbol{\omega}) \leq \cdots \leq H_{n_q-1}^{(q)}(\boldsymbol{\omega}) < 2\pi$$

Sketch of the proof

- **Reformulation** of the problem on the unit circle $\mathbb{S}^1 \subset \mathbb{C}$

$[z,\, z']_{\mathbb{S}^1} \subset \mathbb{S}^1$ arc on the unit circle going from z to z'

$\boldsymbol{h_p} = ms\boldsymbol{p}$

$Z_{\mathrm{X}} : \boldsymbol{x} \mapsto \mathrm{e}^{i\, H_{\mathrm{X}}(\boldsymbol{x})}$ and $Z_{\boldsymbol{p}} : \boldsymbol{\omega} \mapsto = \mathrm{e}^{i\langle \boldsymbol{\omega},\, \boldsymbol{p}\rangle}$

$\boldsymbol{\nu} = \boldsymbol{\theta}/s$

- $G_{\mathrm{X}}\big(\boldsymbol{x_n},\, \boldsymbol{h_p}\big) = \mathrm{Re}\big(Z_{\mathrm{X}}^*(\boldsymbol{x_n})\, Z_{\boldsymbol{p}}(m\boldsymbol{\theta})\big)$ → $\cos\big(\langle \boldsymbol{\nu},\, \boldsymbol{h_p}\rangle - H(\boldsymbol{x_n})\big)$

- Sort $\{Z_{\boldsymbol{p}}(\boldsymbol{\omega})\}_{\boldsymbol{p}\in\{-q..q\}^2}$ → $\big(Z_i^{(q)}(\boldsymbol{\omega})\big)_{i\in\{0..n_q-1\}}$ $\quad n_q := (2q+1)^2$

in ascending order of their argument:

$$0 = H_0^{(q)}(\boldsymbol{\omega}) \leq \cdots \leq H_{n_q-1}^{(q)}(\boldsymbol{\omega}) < 2\pi$$

$H_{n_q}^{(q)}(\boldsymbol{\omega}) := 2\pi$

$Z_{n_q}^{(q)}(\boldsymbol{\omega}) := 1$

Sketch of the proof

- **Reformulation** of the problem on the unit circle $\mathbb{S}^1 \subset \mathbb{C}$

$[z,\, z']_{\mathbb{S}^1} \subset \mathbb{S}^1$ arc on the unit circle going from z to z'

$$\boldsymbol{h_p} = ms\boldsymbol{p}$$

$$Z_{\mathrm{X}} : \boldsymbol{x} \mapsto \mathrm{e}^{i\, H_{\mathrm{X}}(\boldsymbol{x})} \qquad \text{and} \qquad Z_{\boldsymbol{p}} : \boldsymbol{\omega} \mapsto = \mathrm{e}^{i\langle \boldsymbol{\omega},\, \boldsymbol{p}\rangle}$$

$$\boldsymbol{\nu} = \boldsymbol{\theta}/s$$

- $G_{\mathrm{X}}\big(\boldsymbol{x_n},\, \boldsymbol{h_p}\big) = \mathrm{Re}\big(Z_{\mathrm{X}}^*(\boldsymbol{x_n})\, Z_{\boldsymbol{p}}(m\boldsymbol{\theta})\big) \longrightarrow \cos\big(\langle \boldsymbol{\nu},\, \boldsymbol{h_p}\rangle - H(\boldsymbol{x_n})\big)$
- Sort $\{Z_{\boldsymbol{p}}(\boldsymbol{\omega})\}_{\boldsymbol{p}\in\{-q..q\}^2} \longrightarrow \big(Z_i^{(q)}(\boldsymbol{\omega})\big)_{i\in\{0..n_q-1\}}$ $\qquad n_q := (2q+1)^2$

  in ascending order of their argument:

$$0 = H_0^{(q)}(\boldsymbol{\omega}) \leq \cdots \leq H_{n_q-1}^{(q)}(\boldsymbol{\omega}) < 2\pi$$

$$H_{n_q}^{(q)}(\boldsymbol{\omega}) := 2\pi \qquad Z_{n_q}^{(q)}(\boldsymbol{\omega}) := 1$$

- Split $\mathbb{S}^1$ into n_q arcs delimited by the $Z_i^{(q)}(\boldsymbol{\omega})$

$$\mathfrak{A}_i^{(q)}(\boldsymbol{\omega}) := \begin{cases} \Big[Z_i^{(q)}(\boldsymbol{\omega}),\, Z_{i+1}^{(q)}(\boldsymbol{\omega})\Big]_{\mathbb{S}^1} & \text{if } H_{i+1}^{(q)}(\boldsymbol{\omega}) - H_i^{(q)}(\boldsymbol{\omega}) < 2\pi; \\ \mathbb{S}^1 & \text{otherwise.} \end{cases}$$

Sketch of the proof

- **Reformulation** of the problem on the unit circle $\mathbb{S}^1 \subset \mathbb{C}$

$[z,\, z']_{\mathbb{S}^1} \subset \mathbb{S}^1$ arc on the unit circle going from z to z'

$$\boldsymbol{h_p} = ms\boldsymbol{p}$$

$$Z_{\mathrm{X}} : \boldsymbol{x} \mapsto \mathrm{e}^{i\, H_{\mathrm{X}}(\boldsymbol{x})} \qquad \text{and} \qquad Z_{\boldsymbol{p}} : \boldsymbol{\omega} \mapsto = \mathrm{e}^{i\langle \boldsymbol{\omega},\, \boldsymbol{p}\rangle}$$

$$\boldsymbol{\nu} = \boldsymbol{\theta}/s$$

- $G_{\mathrm{X}}\big(\boldsymbol{x_n},\, \boldsymbol{h_p}\big) = \mathrm{Re}\big(Z_{\mathrm{X}}^*(\boldsymbol{x_n})\, Z_{\boldsymbol{p}}(m\boldsymbol{\theta})\big) \longrightarrow \cos\big(\langle \boldsymbol{\nu},\, \boldsymbol{h_p}\rangle - H(\boldsymbol{x_n})\big)$
- Sort $\{Z_{\boldsymbol{p}}(\boldsymbol{\omega})\}_{\boldsymbol{p}\in\{-q..q\}^2} \longrightarrow \big(Z_i^{(q)}(\boldsymbol{\omega})\big)_{i\in\{0..n_q-1\}}$ $\qquad n_q := (2q+1)^2$

  in ascending order of their argument:

$$0 = H_0^{(q)}(\boldsymbol{\omega}) \leq \cdots \leq H_{n_q-1}^{(q)}(\boldsymbol{\omega}) < 2\pi$$

$$H_{n_q}^{(q)}(\boldsymbol{\omega}) := 2\pi \qquad Z_{n_q}^{(q)}(\boldsymbol{\omega}) := 1$$

- Split $\mathbb{S}^1$ into n_q arcs delimited by the $Z_i^{(q)}(\boldsymbol{\omega})$

$$\delta H_i^{(q)}(\boldsymbol{\omega})$$

$$\mathfrak{A}_i^{(q)}(\boldsymbol{\omega}) := \begin{cases} \Big[Z_i^{(q)}(\boldsymbol{\omega}),\, Z_{i+1}^{(q)}(\boldsymbol{\omega})\Big]_{\mathbb{S}^1} & \text{if } H_{i+1}^{(q)}(\boldsymbol{\omega}) - H_i^{(q)}(\boldsymbol{\omega}) < 2\pi; \\ \mathbb{S}^1 & \text{otherwise.} \end{cases}$$

Sketch of the proof

- **Reformulation** of the problem on the unit circle $\mathbb{S}^1 \subset \mathbb{C}$

  $[z,\, z']_{\mathbb{S}^1} \subset \mathbb{S}^1$ arc on the unit circle going from z to z'

  $$\boldsymbol{h_p} = ms\boldsymbol{p}$$

  $$Z_{\mathrm{X}} : \boldsymbol{x} \mapsto \mathrm{e}^{i\, H_{\mathrm{X}}(\boldsymbol{x})} \qquad \text{and} \qquad Z_{\boldsymbol{p}} : \boldsymbol{\omega} \mapsto = \mathrm{e}^{i \langle \boldsymbol{\omega},\, \boldsymbol{p} \rangle}$$

  $$\boldsymbol{\nu} = \boldsymbol{\theta}/s$$

- $G_{\mathrm{X}}\big(\boldsymbol{x_n},\, \boldsymbol{h_p}\big) = \mathrm{Re}\big(Z^*_{\mathrm{X}}(\boldsymbol{x_n})\, Z_{\boldsymbol{p}}(m\boldsymbol{\theta})\big)$ ⟶ $\cos\big(\langle \boldsymbol{\nu},\, \boldsymbol{h_p} \rangle - H(\boldsymbol{x_n})\big)$

Sketch of the proof

- **Reformulation** of the problem on the unit circle $\mathbb{S}^1 \subset \mathbb{C}$

  $[z,\, z']_{\mathbb{S}^1} \subset \mathbb{S}^1$ arc on the unit circle going from z to z' $\boldsymbol{h_p} = ms\boldsymbol{p}$

  $Z_{\mathrm{X}} : \boldsymbol{x} \mapsto \mathrm{e}^{i\, H_{\mathrm{X}}(\boldsymbol{x})}$ and $Z_{\boldsymbol{p}} : \boldsymbol{\omega} \mapsto= \mathrm{e}^{i\langle \boldsymbol{\omega},\, \boldsymbol{p}\rangle}$ $\boldsymbol{\nu} = \boldsymbol{\theta}/s$

- $G_{\mathrm{X}}\big(\boldsymbol{x_n},\, \boldsymbol{h_p}\big) = \mathrm{Re}\big(Z^*_{\mathrm{X}}(\boldsymbol{x_n})\, Z_{\boldsymbol{p}}(m\boldsymbol{\theta})\big)$ ⟶ $\cos\big(\langle \boldsymbol{\nu},\, \boldsymbol{h_p}\rangle - H(\boldsymbol{x_n})\big)$
- $\mathsf{G}^{\max}_{\mathrm{X}}(\boldsymbol{x}) = g_{\max}(\mathsf{Z}_{\mathrm{X}}(\boldsymbol{x}))$ ⟶ $g_{\max} : z \mapsto \max_{\|\boldsymbol{p}\|_\infty \leq q} \mathrm{Re}\big(z^* Z_{\boldsymbol{p}}\big)$

Sketch of the proof

- **Reformulation** of the problem on the unit circle $\mathbb{S}^1 \subset \mathbb{C}$

  $[z,\, z']_{\mathbb{S}^1} \subset \mathbb{S}^1$ arc on the unit circle going from z to z'

  $$\boldsymbol{h}_{\boldsymbol{p}} = ms\boldsymbol{p}$$

  $$Z_{\mathrm{X}} : \boldsymbol{x} \mapsto \mathrm{e}^{i\, H_{\mathrm{X}}(\boldsymbol{x})} \qquad \text{and} \qquad Z_{\boldsymbol{p}} : \boldsymbol{\omega} \mapsto = \mathrm{e}^{i\langle \boldsymbol{\omega},\, \boldsymbol{p}\rangle}$$

  $$\boldsymbol{\nu} = \boldsymbol{\theta}/s$$

- $G_{\mathrm{X}}\big(\boldsymbol{x}_{\boldsymbol{n}},\, \boldsymbol{h}_{\boldsymbol{p}}\big) = \mathrm{Re}\big(Z_{\mathrm{X}}^*(\boldsymbol{x}_{\boldsymbol{n}})\, Z_{\boldsymbol{p}}(m\boldsymbol{\theta})\big)$ ⟶ $\cos\big(\langle \boldsymbol{\nu},\, \boldsymbol{h}_{\boldsymbol{p}}\rangle - H(\boldsymbol{x}_{\boldsymbol{n}})\big)$
- $\mathsf{G}_{\mathrm{X}}^{\max}(\boldsymbol{x}) = g_{\max}(\mathsf{Z}_{\mathrm{X}}(\boldsymbol{x}))$ ⟶ $g_{\max} : z \mapsto \max\limits_{\|\boldsymbol{p}\|_\infty \leq q} \mathrm{Re}\big(z^* Z_{\boldsymbol{p}}\big)$
- $\mathsf{Z}_{\mathrm{X}}(\boldsymbol{x})$ **uniformly distributed** on the unit circle (Hypothesis)

Sketch of the proof

- **Reformulation** of the problem on the unit circle $\mathbb{S}^1 \subset \mathbb{C}$

$[z,\, z']_{\mathbb{S}^1} \subset \mathbb{S}^1$ arc on the unit circle going from z to z'

$\boldsymbol{h_p} = ms\boldsymbol{p}$

$Z_{\mathrm{X}} : \boldsymbol{x} \mapsto \mathrm{e}^{i\, H_{\mathrm{X}}(\boldsymbol{x})}$ and $Z_{\boldsymbol{p}} : \boldsymbol{\omega} \mapsto = \mathrm{e}^{i\langle \boldsymbol{\omega},\, \boldsymbol{p}\rangle}$

$\boldsymbol{\nu} = \boldsymbol{\theta}/s$

- $G_{\mathrm{X}}\big(\boldsymbol{x_n},\, \boldsymbol{h_p}\big) = \mathrm{Re}\big(Z_{\mathrm{X}}^*(\boldsymbol{x_n})\, Z_{\boldsymbol{p}}(m\boldsymbol{\theta})\big)$ ⟶ $\cos\big(\langle \boldsymbol{\nu},\, \boldsymbol{h_p}\rangle - H(\boldsymbol{x_n})\big)$
- $\mathsf{G}_{\mathrm{X}}^{\max}(\boldsymbol{x}) = g_{\max}(\mathsf{Z}_{\mathrm{X}}(\boldsymbol{x}))$ ⟶ $g_{\max} : z \mapsto \max_{\|\boldsymbol{p}\|_\infty \leq q} \mathrm{Re}\big(z^* Z_{\boldsymbol{p}}\big)$
- $\mathsf{Z}_{\mathrm{X}}(\boldsymbol{x})$ **uniformly distributed** on the unit circle (Hypothesis)
- The p-th moment is given by

$$\mathbb{E}\left[\mathsf{G}_{\mathrm{X}}^{\max}(\boldsymbol{x})^p\right] = \frac{1}{2\pi}\int_{\mathbb{S}^1} g_{\max}(z)^p\, \mathrm{d}\vartheta(z) = \frac{1}{2\pi}\sum_{i=0}^{n_q-1}\int_{\mathfrak{A}_i^{(q)}} g_{\max}(z)^p\, \mathrm{d}\vartheta(z).$$

Sketch of the proof

- **Reformulation** of the problem on the unit circle $\mathbb{S}^1 \subset \mathbb{C}$

  $[z,\, z']_{\mathbb{S}^1} \subset \mathbb{S}^1$ arc on the unit circle going from z to z'

  $\boldsymbol{h_p} = ms\boldsymbol{p}$

  $Z_{\mathrm{X}} : \boldsymbol{x} \mapsto \mathrm{e}^{i\, H_{\mathrm{X}}(\boldsymbol{x})}$ and $Z_{\boldsymbol{p}} : \boldsymbol{\omega} \mapsto = \mathrm{e}^{i\langle \boldsymbol{\omega},\, \boldsymbol{p}\rangle}$

  $\boldsymbol{\nu} = \boldsymbol{\theta}/s$

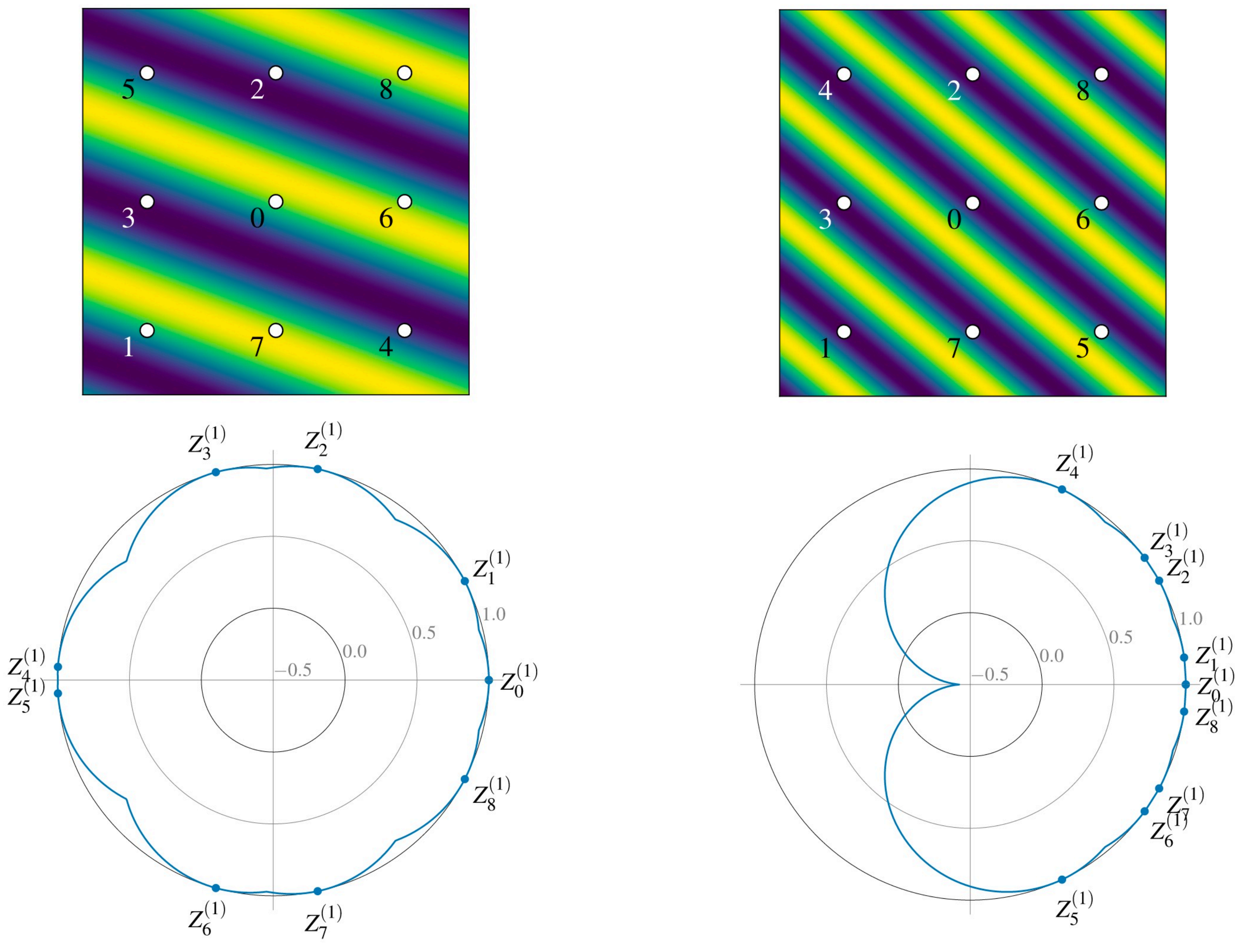
- $G_{\mathrm{X}}\big(\boldsymbol{x_n},\, \boldsymbol{h_p}\big) = \mathrm{Re}\big(Z^*_{\mathrm{X}}(\boldsymbol{x_n})\, Z_{\boldsymbol{p}}(m\boldsymbol{\theta})\big)$ ⟶ $\cos\big(\langle \boldsymbol{\nu},\, \boldsymbol{h_p}\rangle - H(\boldsymbol{x_n})\big)$
- $\mathsf{G}^{\max}_{\mathrm{X}}(\boldsymbol{x}) = g_{\max}(\mathsf{Z}_{\mathrm{X}}(\boldsymbol{x}))$ ⟶ $g_{\max} : z \mapsto \max_{\|\boldsymbol{p}\|_\infty \le q} \mathrm{Re}\big(z^* Z_{\boldsymbol{p}}\big)$
- $\mathsf{Z}_{\mathrm{X}}(\boldsymbol{x})$ **uniformly distributed** on the unit circle (Hypothesis)
- The p-th moment is given by

$$\mathbb{E}\left[\mathsf{G}^{\max}_{\mathrm{X}}(\boldsymbol{x})^p\right] = \frac{1}{2\pi}\int_{\mathbb{S}^1} g_{\max}(z)^p \,\mathrm{d}\vartheta(z) = \frac{1}{2\pi}\sum_{i=0}^{n_q-1}\int_{\mathfrak{A}_i^{(q)}} g_{\max}(z)^p \,\mathrm{d}\vartheta(z).$$

- $\forall z \in \mathfrak{A}_i^{(q)},\ g_{\max}(z) = \max\left(\mathrm{Re}\big(z^* Z_i^{(q)}\big),\, \mathrm{Re}\big(z^* Z_{i+1}^{(q)}\big)\right)$

Sketch of the proof

$$g_{\max} : z \mapsto \max_{\|\boldsymbol{p}\|_\infty \leq q} \mathrm{Re}\big(z^* Z_{\boldsymbol{p}}\big)$$



(a) General case (b) Pathological case

Sketch of the proof

- **Reformulation** of the problem on the unit circle $\mathbb{S}^1 \subset \mathbb{C}$
  - $\forall z \in \mathfrak{A}_i^{(q)},\ g_{\max}(z) = \max\Big(\mathrm{Re}\big(z^* Z_i^{(q)}\big),\ \mathrm{Re}\big(z^* Z_{i+1}^{(q)}\big)\Big)$

Sketch of the proof

- **Reformulation** of the problem on the unit circle $\mathbb{S}^1 \subset \mathbb{C}$
  - $\forall z \in \mathfrak{A}_i^{(q)},\ g_{\max}(z) = \max\Big(\mathrm{Re}\big(z^* Z_i^{(q)}\big),\ \mathrm{Re}\big(z^* Z_{i+1}^{(q)}\big)\Big)$
  - $\forall z \in \overline{\mathfrak{A}}_i^{(q)},\ g_{\max}(z) = \mathrm{Re}\big(z^* Z_i^{(q)}\big)$

Sketch of the proof

- **Reformulation** of the problem on the unit circle $\mathbb{S}^1 \subset \mathbb{C}$
  - $\forall z \in \mathfrak{A}_i^{(q)},\ g_{\max}(z) = \max\left(\mathrm{Re}\big(z^* Z_i^{(q)}\big),\ \mathrm{Re}\big(z^* Z_{i+1}^{(q)}\big)\right)$
  - $\forall z \in \overline{\mathfrak{A}}_i^{(q)},\ g_{\max}(z) = \mathrm{Re}\big(z^* Z_i^{(q)}\big)$

$$
\begin{aligned}
\int_{\mathfrak{A}_i^{(q)}} g_{\max}(z)^p \,\mathrm{d}\vartheta(z) &= 2 \int_{\overline{\mathfrak{A}}_i^{(q)}} \mathrm{Re}\big(z^* Z_i^{(q)}\big)^p \,\mathrm{d}\vartheta(z) \\
&= 2 \int_{H_i^{(q)}}^{\overline{H}_i^{(q)}} \cos^p\big(\eta - H_i^{(q)}\big)\,\mathrm{d}\eta \\
&= 2 \int_0^{\delta H_i^{(q)}/2} \cos^p \eta' \,\mathrm{d}\eta'
\end{aligned}
$$

Sketch of the proof

- **Reformulation** of the problem on the unit circle $\mathbb{S}^1 \subset \mathbb{C}$
  - $\forall z \in \mathfrak{A}_i^{(q)},\ g_{\max}(z) = \max\left(\mathrm{Re}\big(z^* Z_i^{(q)}\big),\ \mathrm{Re}\big(z^* Z_{i+1}^{(q)}\big)\right)$
  - $\forall z \in \overline{\mathfrak{A}}_i^{(q)},\ g_{\max}(z) = \mathrm{Re}\big(z^* Z_i^{(q)}\big)$

$$\begin{aligned}\int_{\mathfrak{A}_i^{(q)}} g_{\max}(z)^p \,\mathrm{d}\vartheta(z) &= 2\int_{\overline{\mathfrak{A}}_i^{(q)}} \mathrm{Re}\big(z^* Z_i^{(q)}\big)^p \,\mathrm{d}\vartheta(z) \\ &= 2\int_{H_i^{(q)}}^{\overline{H}_i^{(q)}} \cos^p\left(\eta - H_i^{(q)}\right) \mathrm{d}\eta \\ &= 2\int_0^{\delta H_i^{(q)}/2} \cos^p \eta' \,\mathrm{d}\eta'\end{aligned}$$

$z \leftarrow \mathrm{e}^{i\eta}$

Sketch of the proof

- **Reformulation** of the problem on the unit circle $\mathbb{S}^1 \subset \mathbb{C}$
  - $\forall z \in \mathfrak{A}_i^{(q)},\ g_{\max}(z) = \max\Big(\mathrm{Re}\big(z^* Z_i^{(q)}\big),\ \mathrm{Re}\big(z^* Z_{i+1}^{(q)}\big)\Big)$
  - $\forall z \in \overline{\mathfrak{A}}_i^{(q)},\ g_{\max}(z) = \mathrm{Re}\big(z^* Z_i^{(q)}\big)$

$$\int_{\mathfrak{A}_i^{(q)}} g_{\max}(z)^p \,\mathrm{d}\vartheta(z) = 2\int_{\overline{\mathfrak{A}}_i^{(q)}} \mathrm{Re}\big(z^* Z_i^{(q)}\big)^p \,\mathrm{d}\vartheta(z)$$

$z \leftarrow \mathrm{e}^{i\eta}$

$$= 2\int_{H_i^{(q)}}^{\overline{H}_i^{(q)}} \cos^p\big(\eta - H_i^{(q)}\big)\,\mathrm{d}\eta$$

$\overline{H}_i^{(q)} := (H_i^{(q)} + H_{i+1}^{(q)})/2$

$$= 2\int_0^{\delta H_i^{(q)}/2} \cos^p \eta' \,\mathrm{d}\eta'$$

Sketch of the proof

- **Reformulation** of the problem on the unit circle $\mathbb{S}^1 \subset \mathbb{C}$
  - $\forall z \in \mathfrak{A}_i^{(q)},\ g_{\max}(z) = \max\Big(\mathrm{Re}\big(z^* Z_i^{(q)}\big),\ \mathrm{Re}\big(z^* Z_{i+1}^{(q)}\big)\Big)$
  - $\forall z \in \overline{\mathfrak{A}}_i^{(q)},\ g_{\max}(z) = \mathrm{Re}\big(z^* Z_i^{(q)}\big)$

$$\begin{aligned}\int_{\mathfrak{A}_i^{(q)}} g_{\max}(z)^p \,\mathrm{d}\vartheta(z) &= 2\int_{\overline{\mathfrak{A}}_i^{(q)}} \mathrm{Re}\big(z^* Z_i^{(q)}\big)^p \,\mathrm{d}\vartheta(z) && z \leftarrow \mathrm{e}^{i\eta} \\ &= 2\int_{H_i^{(q)}}^{\overline{H}_i^{(q)}} \cos^p\big(\eta - H_i^{(q)}\big)\,\mathrm{d}\eta && \overline{H}_i^{(q)} := \big(H_i^{(q)} + H_{i+1}^{(q)}\big)/2 \\ &= 2\int_0^{\delta H_i^{(q)}/2} \cos^p \eta' \,\mathrm{d}\eta' && \eta' \leftarrow \eta - H_i^{(q)}\end{aligned}$$

Sketch of the proof

Sketch of the proof

- $\mathbb{E}\left[\mathsf{G}_{\mathrm{X}}^{\max}(\boldsymbol{x})\right] = \frac{1}{\pi}\sum_{i=0}^{n_q-1} \sin\frac{\delta H_i^{(q)}}{2};$
- $\mathbb{E}\left[\mathsf{G}_{\mathrm{X}}^{\max}(\boldsymbol{x})^2\right] = \frac{1}{2} + \frac{1}{4\pi}\sum_{i=0}^{n_q-1} \sin\delta H_i^{(q)}.$

Sketch of the proof

- $$\mathbb{E}\left[\mathsf{G}_{\mathrm{X}}^{\max}(\boldsymbol{x})\right] = \frac{1}{\pi}\sum_{i=0}^{n_q-1} \sin\frac{\delta H_i^{(q)}}{2};$$
- $$\mathbb{E}\left[\mathsf{G}_{\mathrm{X}}^{\max}(\boldsymbol{x})^2\right] = \frac{1}{2} + \frac{1}{4\pi}\sum_{i=0}^{n_q-1} \sin\delta H_i^{(q)}.$$
- $\mathsf{Q}_{\mathrm{X}} := 1 - \mathsf{G}_{\mathrm{X}}^{\max}$

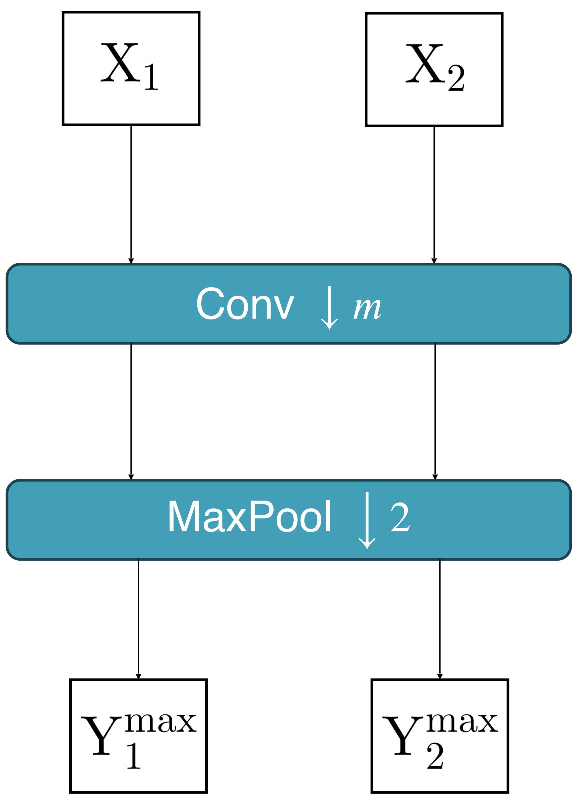
Sketch of the proof

- $$\mathbb{E}\left[\mathsf{G}_{\mathrm{X}}^{\max}(\boldsymbol{x})\right] = \frac{1}{\pi}\sum_{i=0}^{n_q-1} \sin\frac{\delta H_i^{(q)}}{2};$$
- $$\mathbb{E}\left[\mathsf{G}_{\mathrm{X}}^{\max}(\boldsymbol{x})^2\right] = \frac{1}{2} + \frac{1}{4\pi}\sum_{i=0}^{n_q-1} \sin\delta H_i^{(q)}.$$
- $\mathsf{Q}_{\mathrm{X}} := 1 - \mathsf{G}_{\mathrm{X}}^{\max}$
- By linearity of the expected value

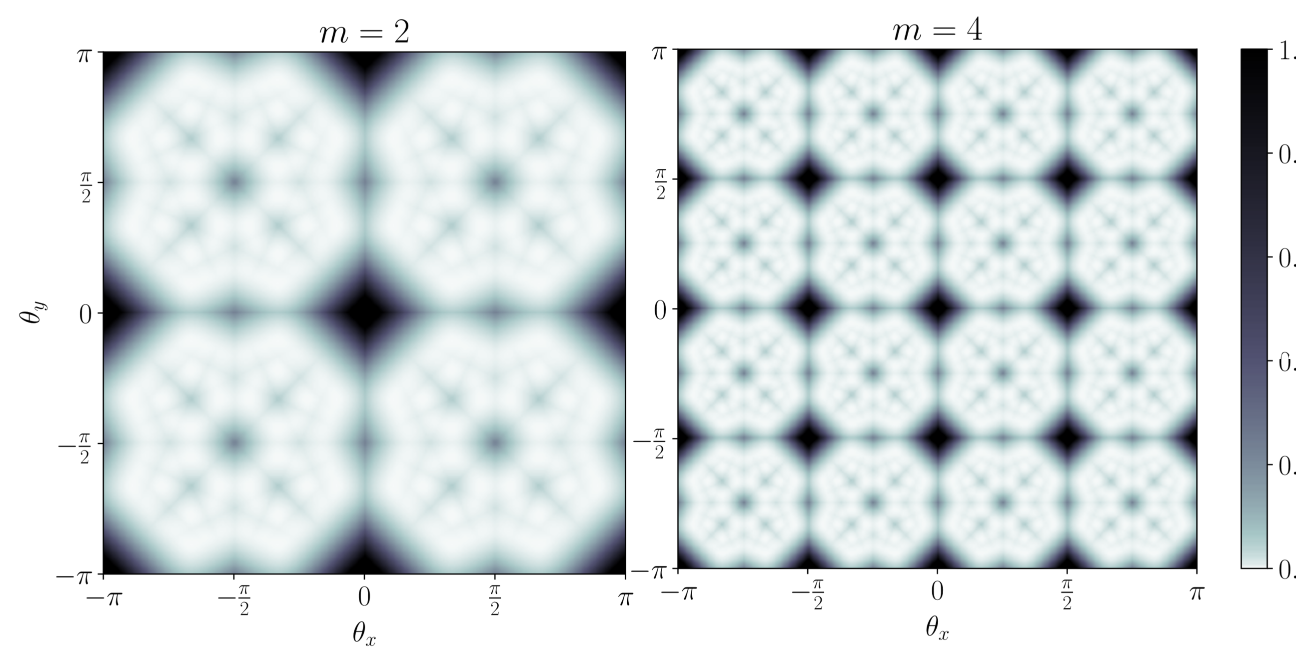
$$\mathbb{E}\left[\mathsf{Q}_{\mathrm{X}}(\boldsymbol{x})^2\right] := \frac{3}{2} + \frac{1}{4\pi}\sum_{i=0}^{n_q-1}\left(\sin\delta H_i^{(q)} - 8\sin\frac{\delta H_i^{(q)}}{2}\right)$$

Main result

- MSE between $\mathbb{C}$**Mod** and $\mathbb{R}$**Max** output

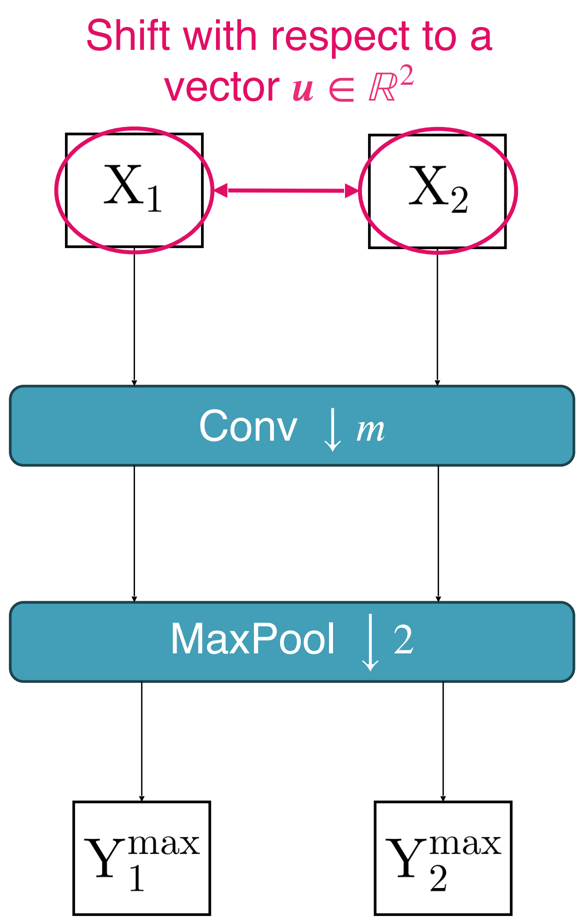


$$\boldsymbol{\theta} \mapsto \gamma_q(m\boldsymbol{\theta})^2$$
$$q = 1$$

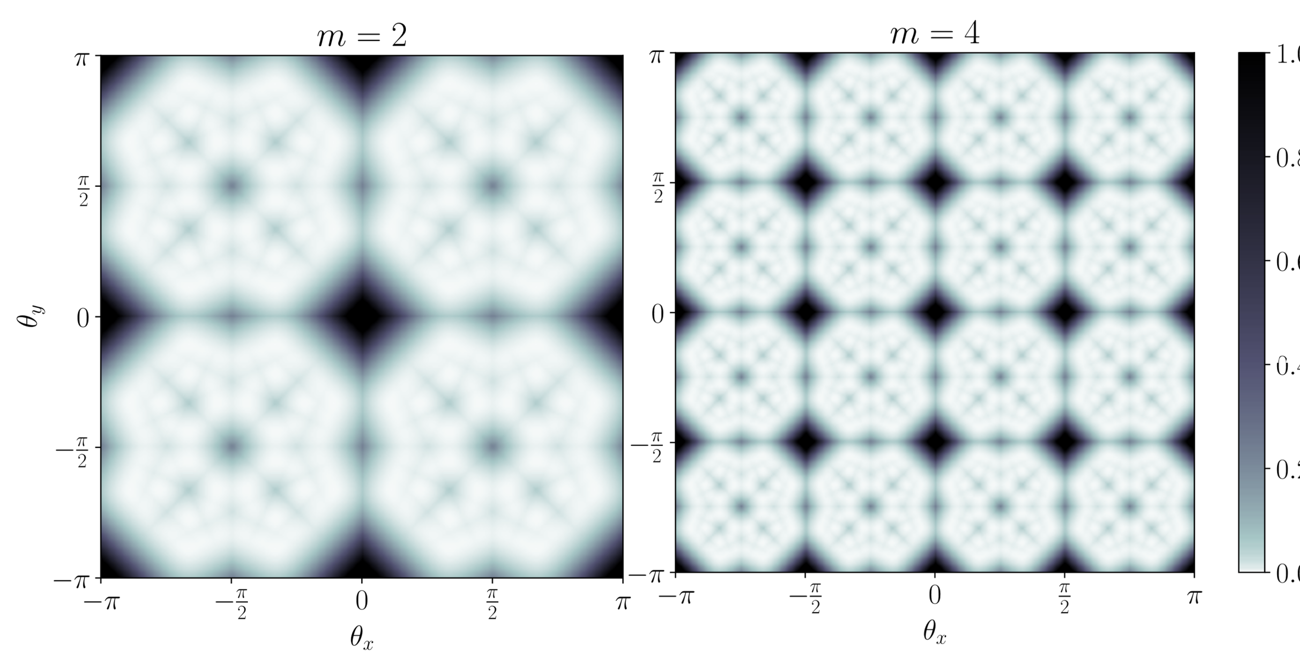


Main result

- MSE between $\mathbb{C}$**Mod** and $\mathbb{R}$**Max** output

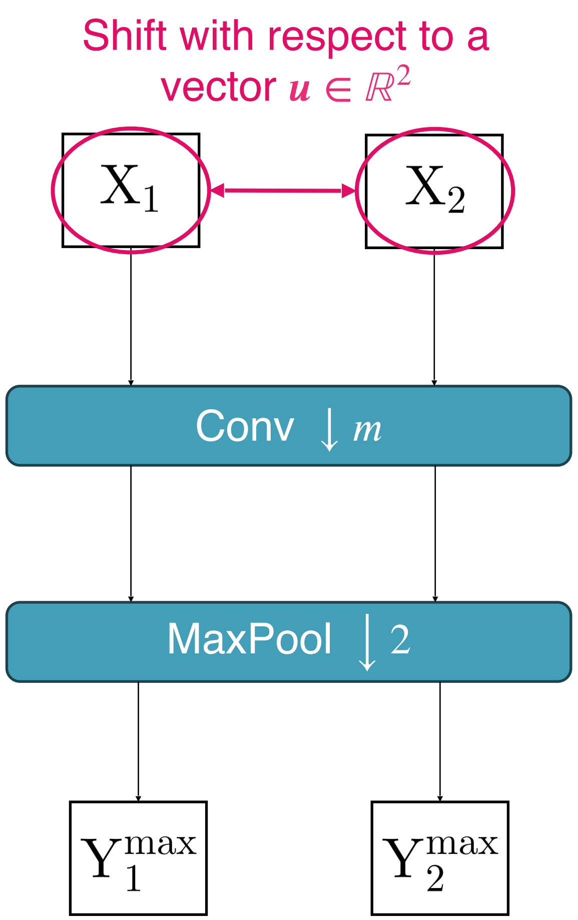


$$\boldsymbol{\theta} \mapsto \gamma_q(m\boldsymbol{\theta})^2$$
$$q = 1$$



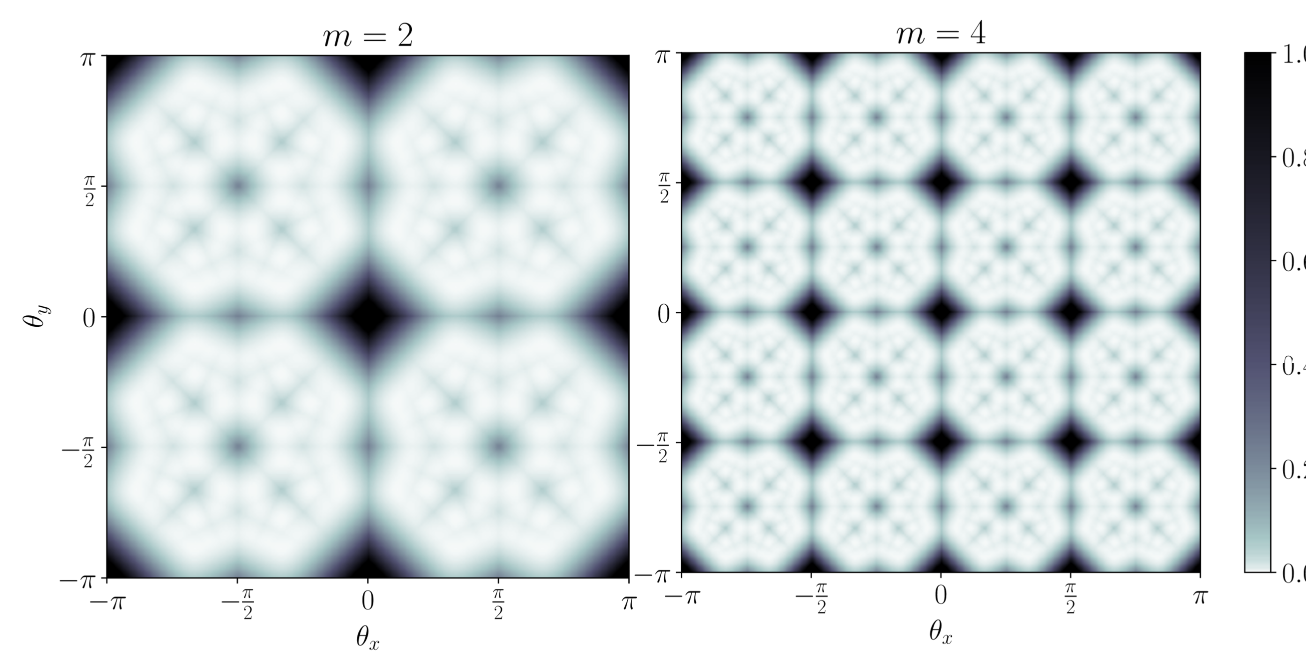
Main result

- MSE between $\mathbb{C}$**Mod** and $\mathbb{R}$**Max** output



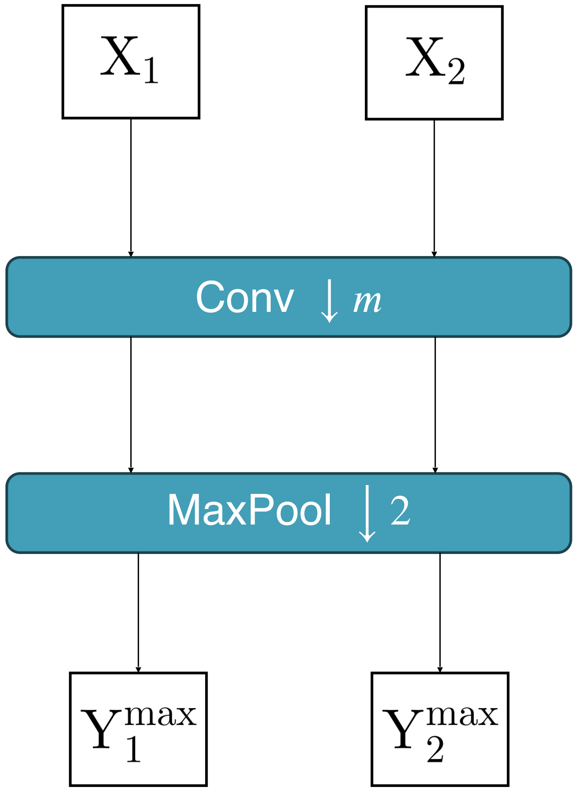
$$\mathbb{E}\left[\frac{\|Y_1^{\max} - Y_2^{\max}\|_2}{\|Y_1^{\mathrm{mod}}\|_2}\right] \leq 2\big(\beta_q(m\kappa) + \gamma_q(m\boldsymbol{\theta})\big) + \alpha(\kappa\boldsymbol{u})$$

$$\boldsymbol{\theta} \mapsto \gamma_q(m\boldsymbol{\theta})^2$$
$$q = 1$$



Main result

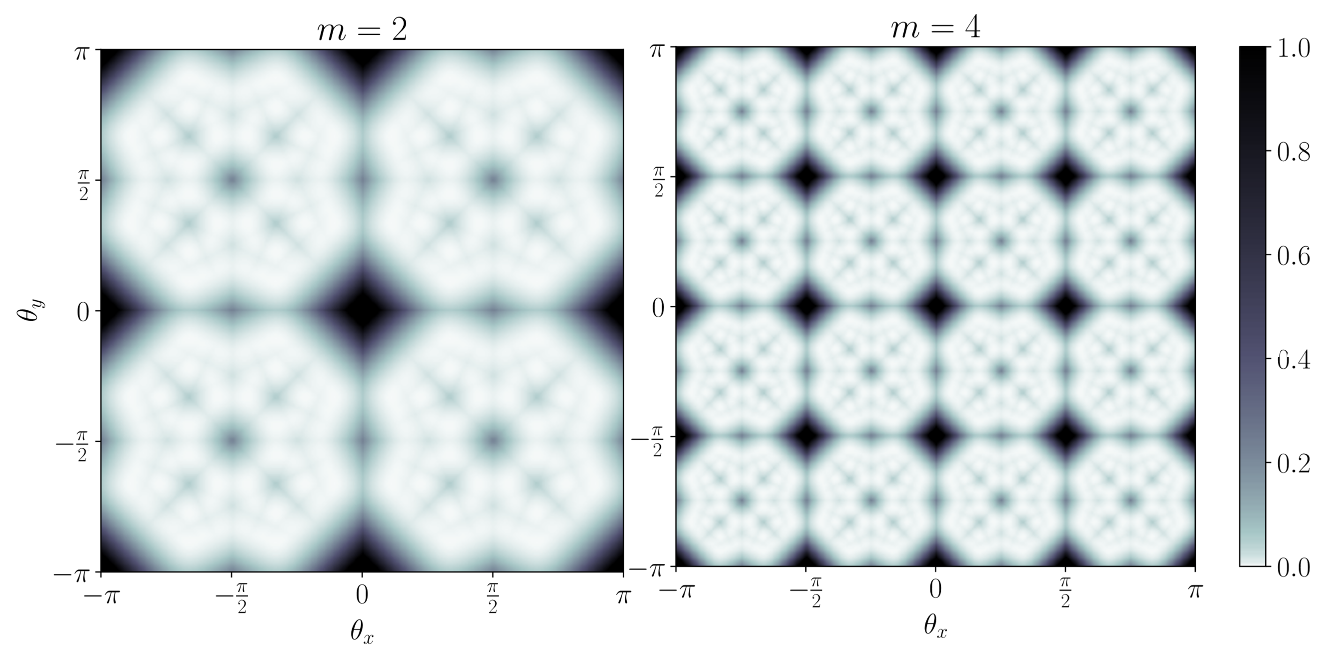
- MSE between $\mathbb{C}$**Mod** and $\mathbb{R}$**Max** output



$$\mathbb{E}\left[\frac{\|Y_1^{\max} - Y_2^{\max}\|_2}{\left\|Y_1^{\mathrm{mod}}\right\|_2}\right] \leq 2\big(\beta_q(m\kappa) + \gamma_q(m\boldsymbol{\theta})\big) + \alpha(\kappa \boldsymbol{u})$$

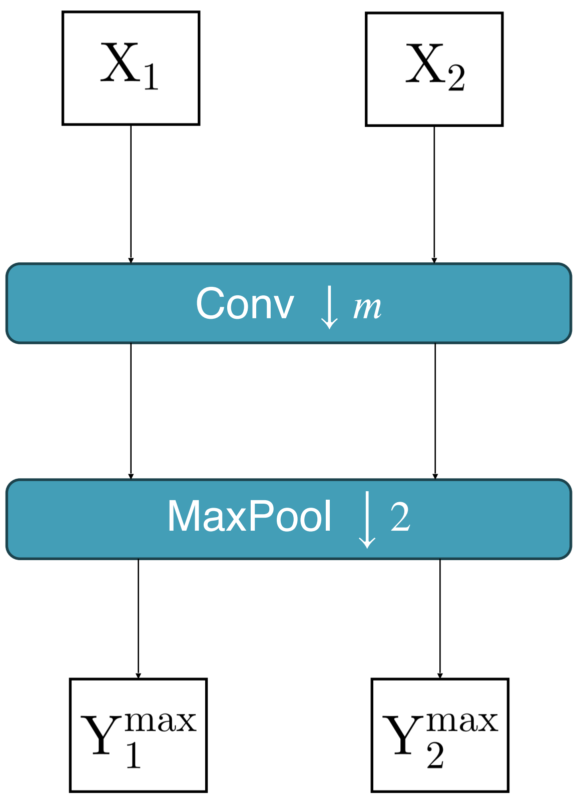
Divergence $\mathbb{R}$Max-$\mathbb{C}$Mod

$$\boldsymbol{\theta} \mapsto \gamma_q(m\boldsymbol{\theta})^2$$
$$q = 1$$



Main result

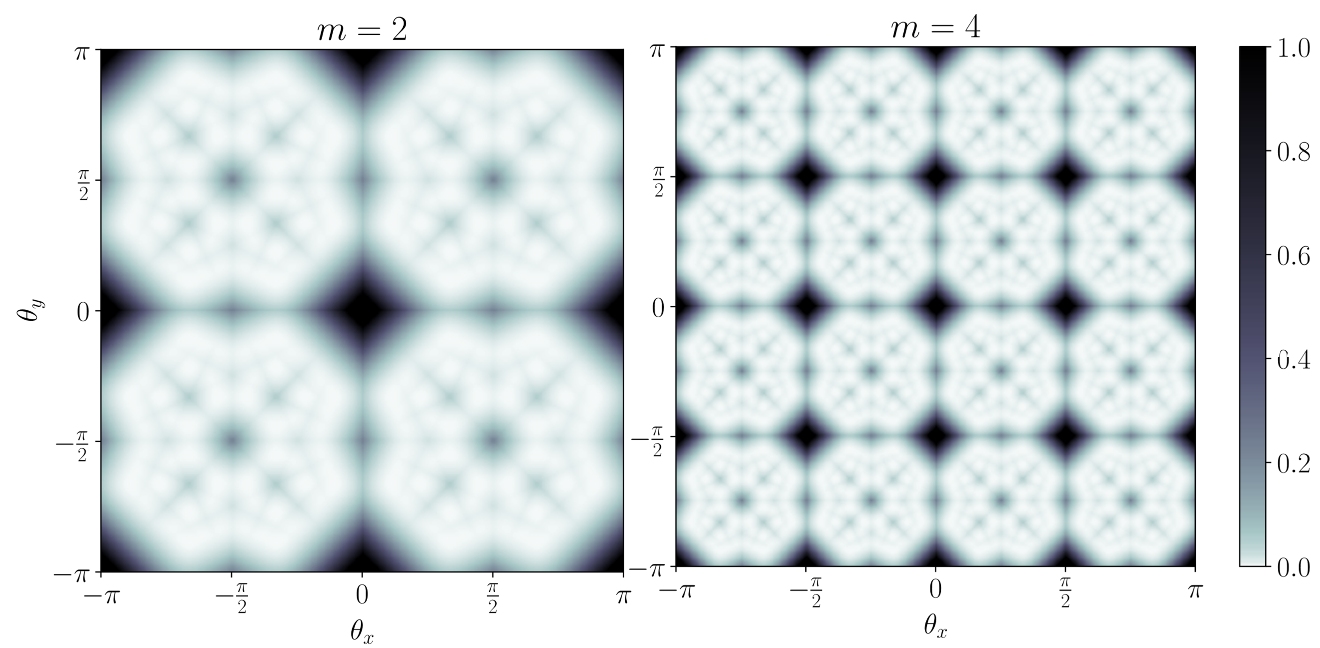
- MSE between $\mathbb{C}$**Mod** and $\mathbb{R}$**Max** output



$$\mathbb{E}\left[\frac{\|\mathrm{Y}_1^{\max} - \mathrm{Y}_2^{\max}\|_2}{\|\mathrm{Y}_1^{\mathrm{mod}}\|_2}\right] \leq 2\big(\beta_q(m\kappa) + \gamma_q(m\boldsymbol{\theta})\big) + \alpha(\kappa\boldsymbol{u})$$

Shift invariance of $\mathbb{C}$Mod ($O(\kappa\boldsymbol{u})$)

$$\boldsymbol{\theta} \mapsto \gamma_q(m\boldsymbol{\theta})^2$$
$$q = 1$$



Experimental validation

Experimental validation

- **What we need:**

Experimental validation

- **What we need:**
  - A fully deterministic model with predefined convolution kernels

Experimental validation

- **What we need:**
  - A fully deterministic model with predefined convolution kernels
  - A set of Gabor-like filters tiling the entire frequency plane and a unique bandwidth share across all filters

Experimental validation

- **What we need:**
  - A fully deterministic model with predefined convolution kernels
  - A set of Gabor-like filters tiling the entire frequency plane and a unique bandwidth share across all filters
  - No prediction, only the first layers are implemented

Experimental validation

- **What we need:**
  - A fully deterministic model with predefined convolution kernels
  - A set of Gabor-like filters tiling the entire frequency plane and a unique bandwidth share across all filters
  - No prediction, only the first layers are implemented
  - Dataset: ImageNet-1K, validation set (50 000 images)

Experimental validation

- **What we need:**
  - A fully deterministic model with predefined convolution kernels
  - A set of Gabor-like filters tiling the entire frequency plane and a unique bandwidth share across all filters
  - No prediction, only the first layers are implemented
  - Dataset: ImageNet-1K, validation set (50 000 images)
- **Proposed solution:**

Experimental validation

- **What we need:**
  - A fully deterministic model with predefined convolution kernels
  - A set of Gabor-like filters tiling the entire frequency plane and a unique bandwidth share across all filters
  - No prediction, only the first layers are implemented
  - Dataset: ImageNet-1K, validation set (50 000 images)
- **Proposed solution:**
  - The dual-tree complex wavelet packet transform (DT-CWPT)

Experimental validation

- **What we need:**
  - A fully deterministic model with predefined convolution kernels
  - A set of Gabor-like filters tiling the entire frequency plane and a unique bandwidth share across all filters
  - No prediction, only the first layers are implemented
  - Dataset: ImageNet-1K, validation set (50 000 images)

- **Proposed solution:**
  - The dual-tree complex wavelet packet transform (DT-CWPT)

I. Bayram and I. W. Selesnick, "On the Dual-Tree Complex Wavelet Packet and M-Band Transforms," IEEE TSP, 2008.

Experimental validation

- **What we need:**
  - A fully deterministic model with predefined convolution kernels
  - A set of Gabor-like filters tiling the entire frequency plane and a unique bandwidth share across all filters
  - No prediction, only the first layers are implemented
  - Dataset: ImageNet-1K, validation set (50 000 images)
- **Proposed solution:**
  - The dual-tree complex wavelet packet transform (DT-CWPT)
  - The bandwidth and subsampling are controlled by the depth J

I. Bayram and I. W. Selesnick, "On the Dual-Tree Complex Wavelet Packet and M-Band Transforms," IEEE TSP, 2008.

Experiments

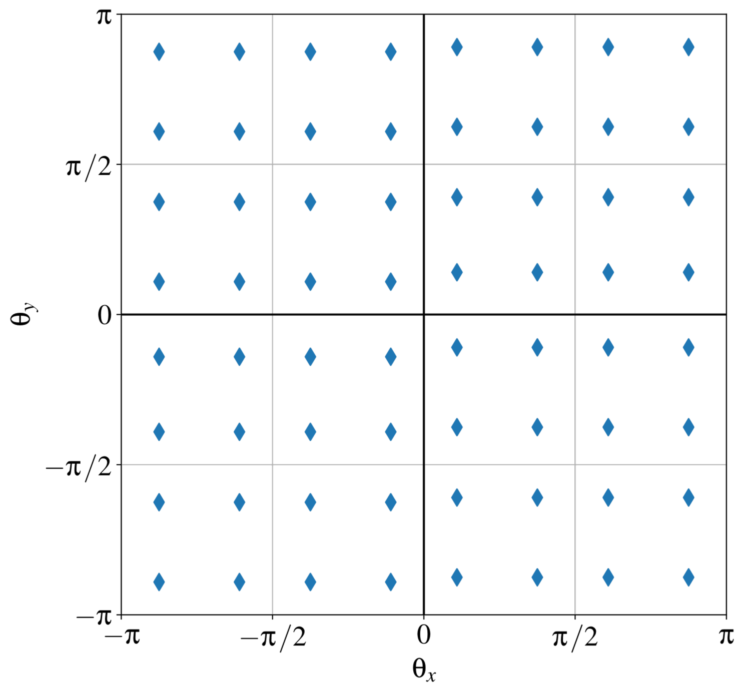
- Filters generated by the DT-CWPT

Case $J = 2$ (two levels of dual-tree decomposition):
$\kappa = \pi/2$;
$m = 2$;
32 filters + complex conjugates.

Experiments

- Filters generated by the DT-CWPT

Case $J = 2$ (two levels of dual-tree decomposition):
$\kappa = \pi/2$;
$m = 2$;
32 filters + complex conjugates.



Experiments

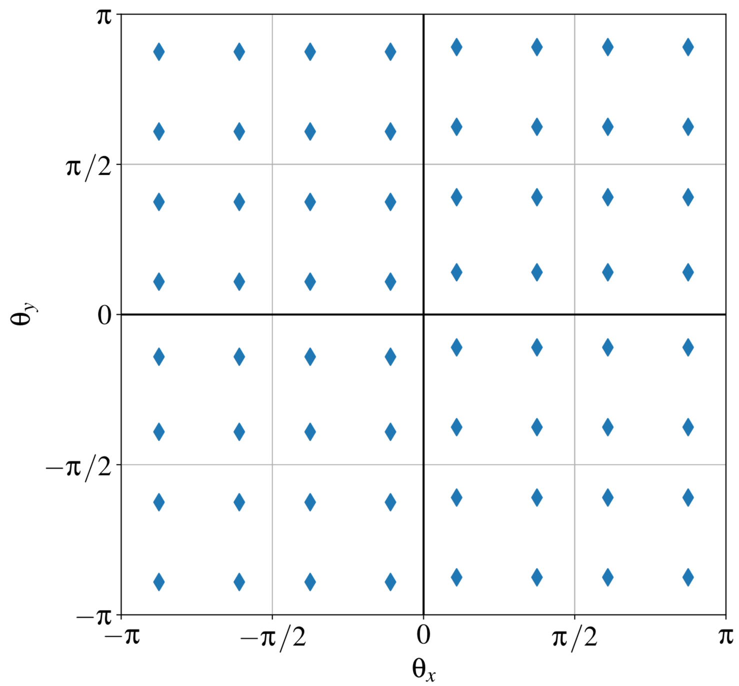
- Filters generated by the DT-CWPT

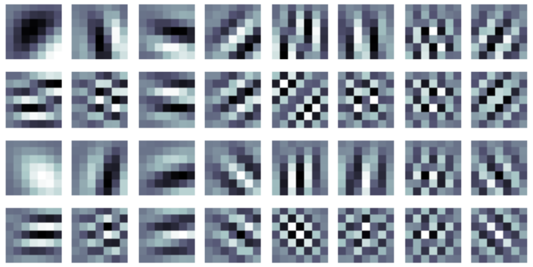
Case $J = 2$ (two levels of dual-tree decomposition):

$\kappa = \pi/2$;

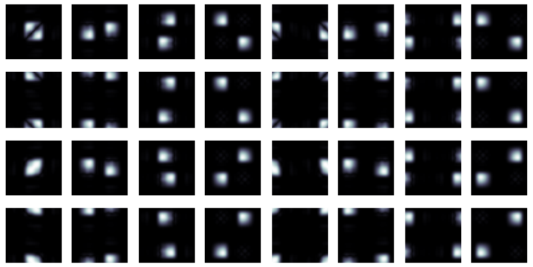
$m = 2$;

32 filters + complex conjugates.





Spatial domain



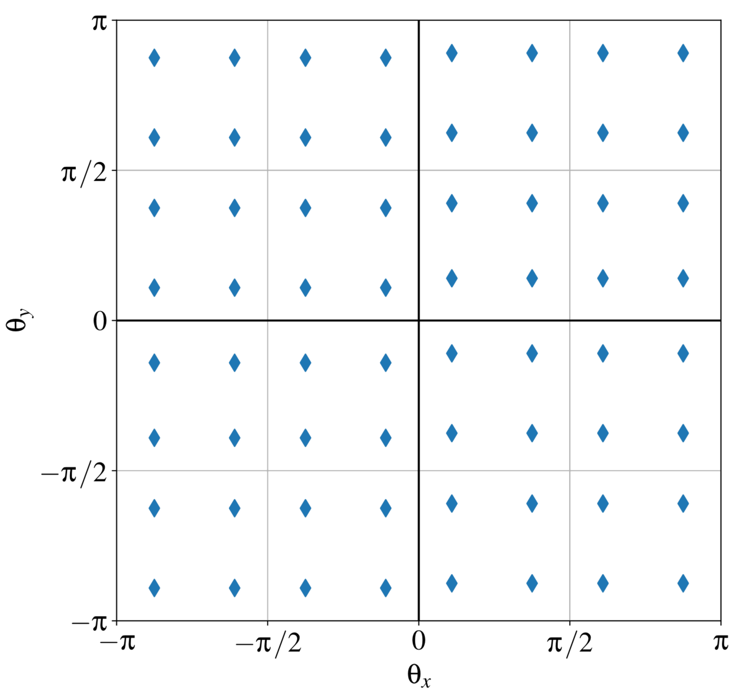
Fourier domain

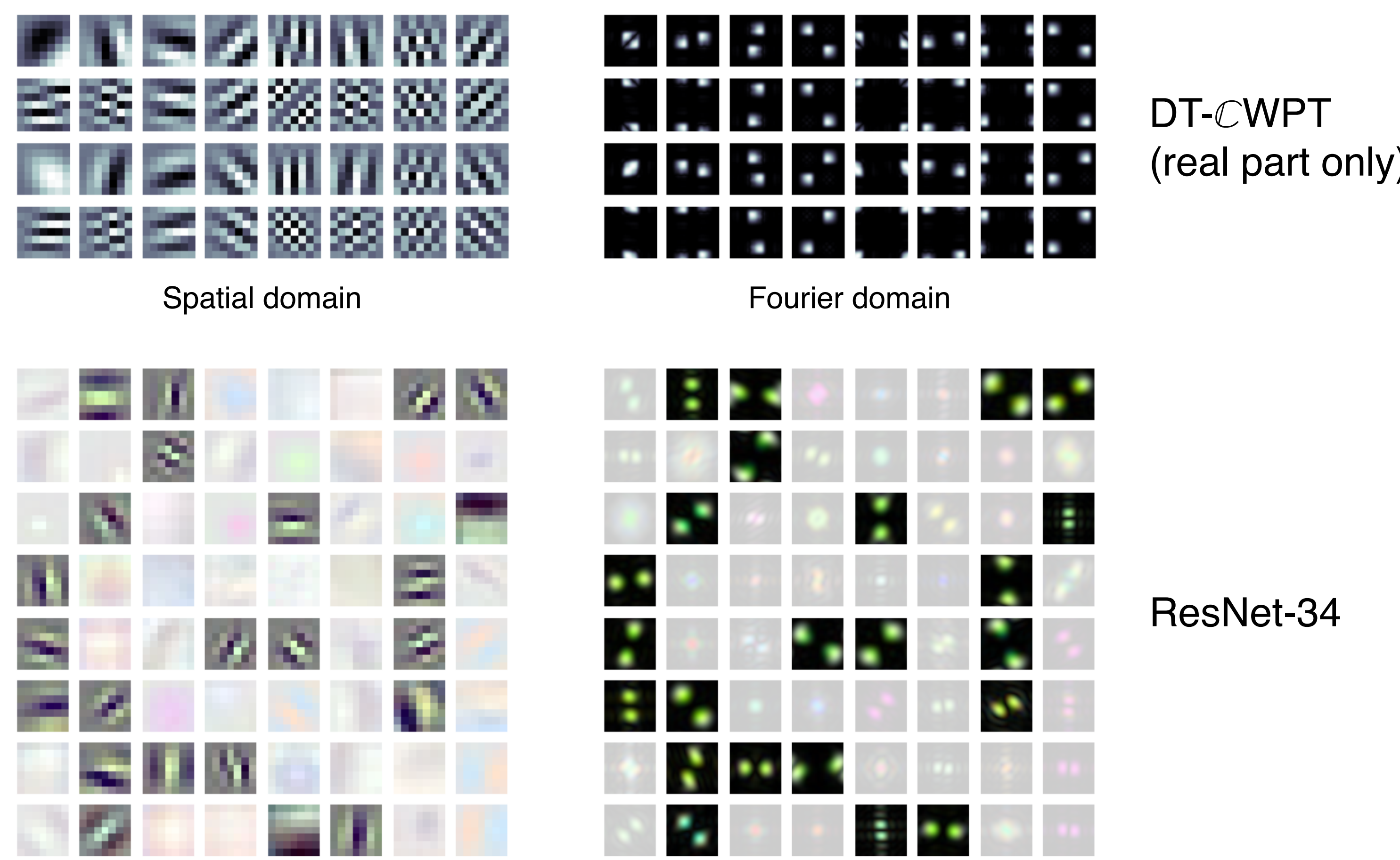
DT-$\mathbb{C}$WPT (real part only)

Experiments

- Filters generated by the DT-CWPT

Case $J = 2$ (two levels of dual-tree decomposition):
$\kappa = \pi/2$;
$m = 2$;
32 filters + complex conjugates.





Experiments

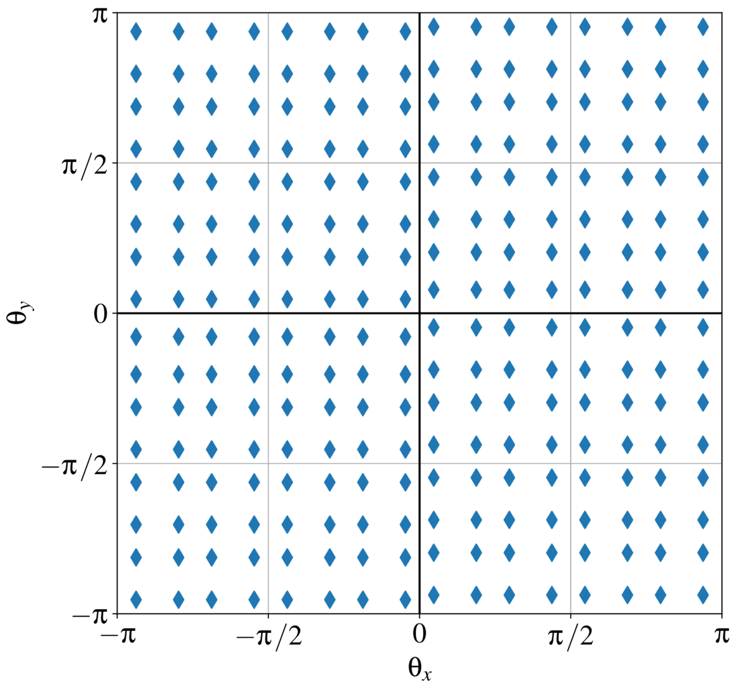
- Filters generated by the DT-CWPT

Case $\boldsymbol{J = 3}$ (three levels of dual-tree decomposition):
$\kappa = \pi/4$;
$m = 4$;
128 filters + complex conjugates.

Experiments

- Filters generated by the DT-CWPT

Case $J = 3$ (three levels of dual-tree decomposition):
$\kappa = \pi/4$;
$m = 4$;
128 filters + complex conjugates.



Experiments

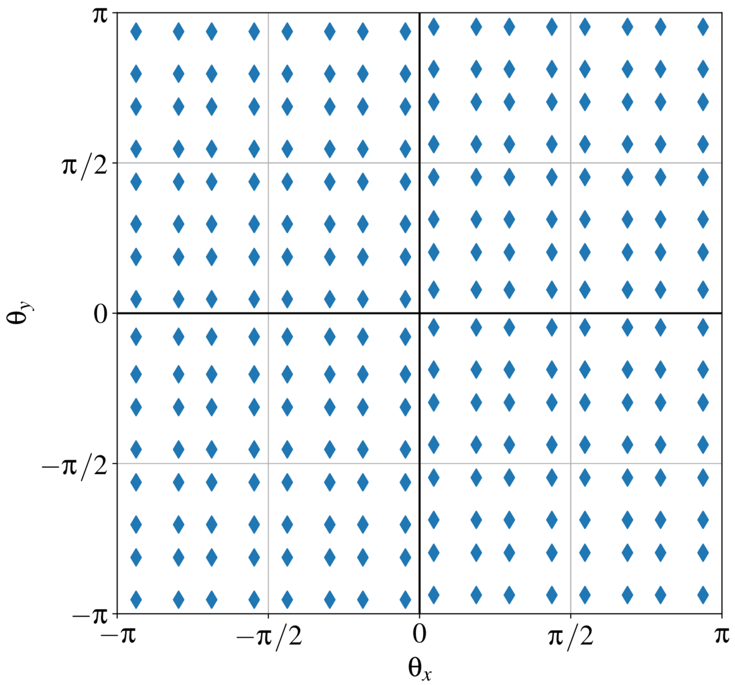
- Filters generated by the DT-CWPT

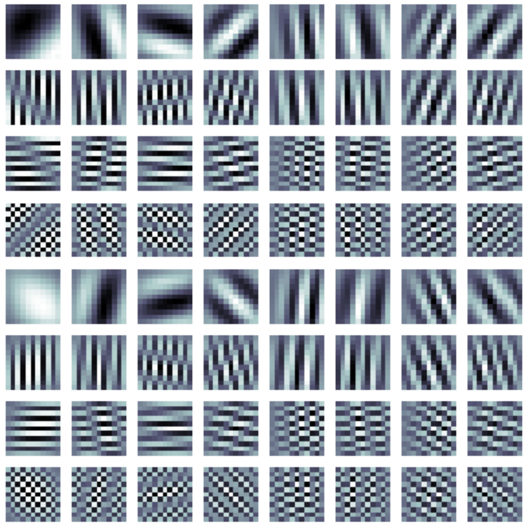
Case $J = 3$ (three levels of dual-tree decomposition):

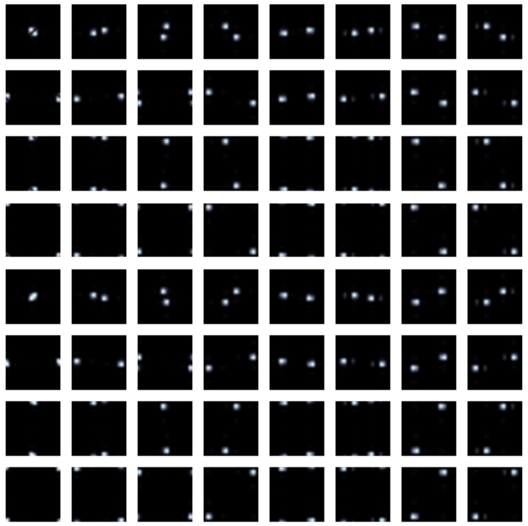
$\kappa = \pi/4$;

$m = 4$;

128 filters + complex conjugates.





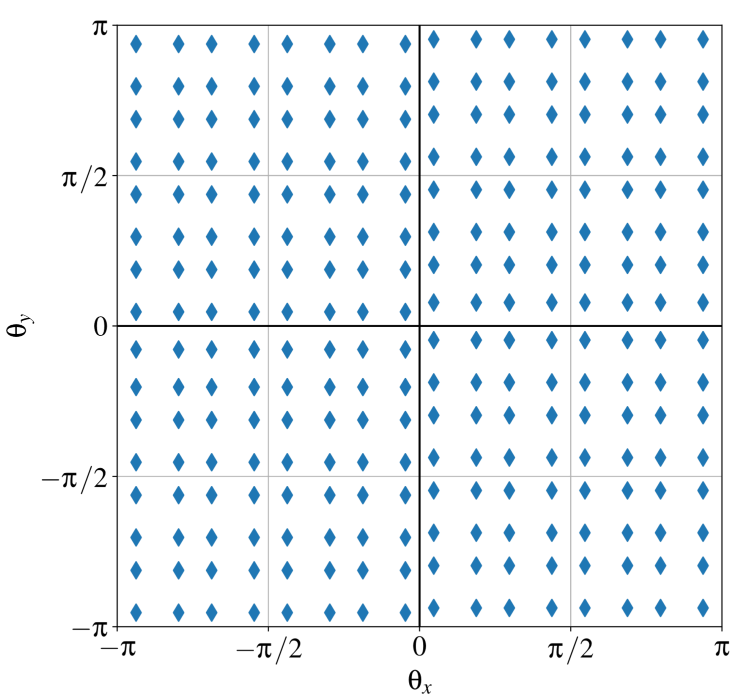


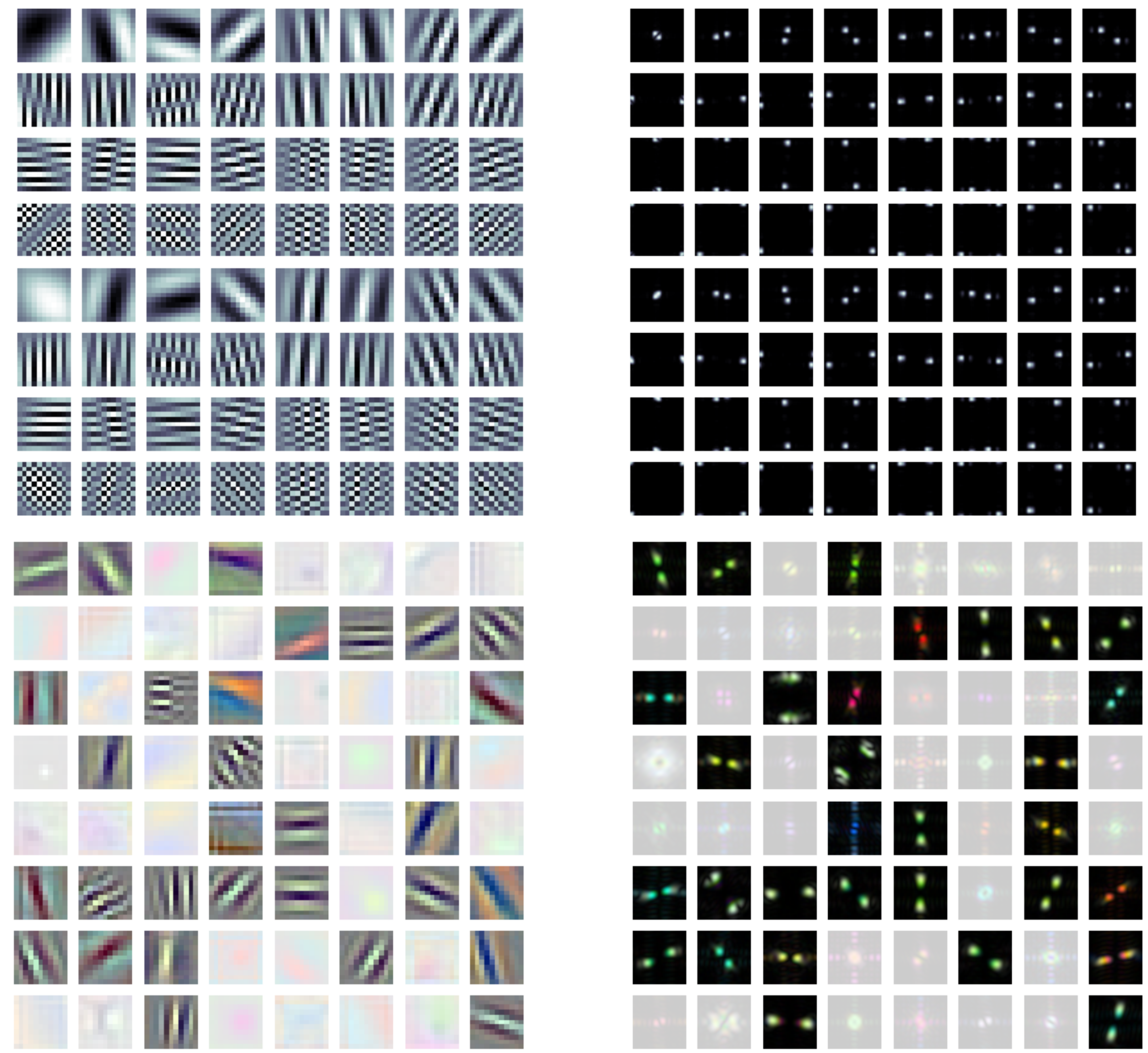
DT-ℂWPT (subset, real part only)

Experiments

- Filters generated by the DT-CWPT

Case $J = 3$ (three levels of dual-tree decomposition):
$\kappa = \pi/4$;
$m = 4$;
128 filters + complex conjugates.





Spatial domain

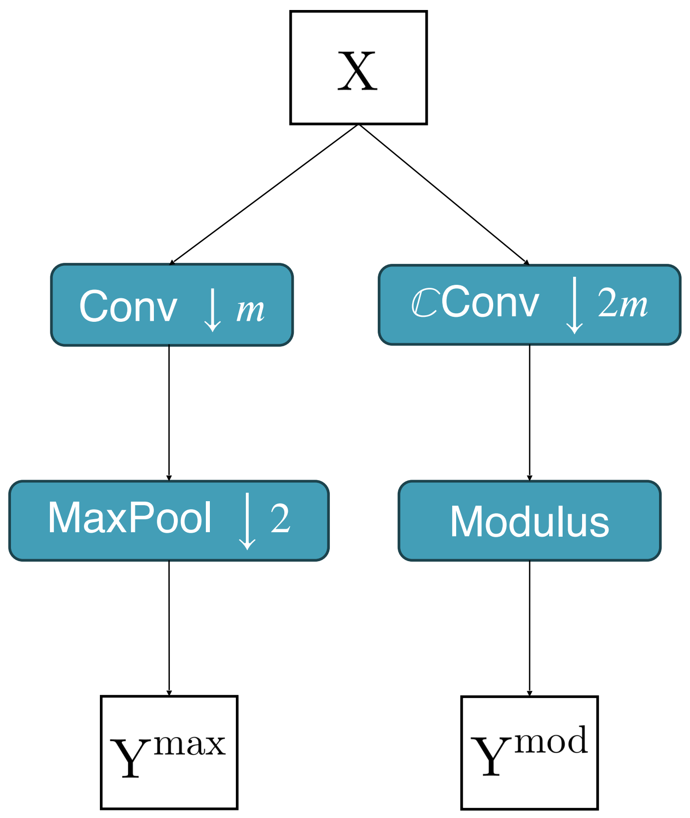
Fourier domain

DT-$\mathbb{C}$WPT (subset, real part only)

AlexNet

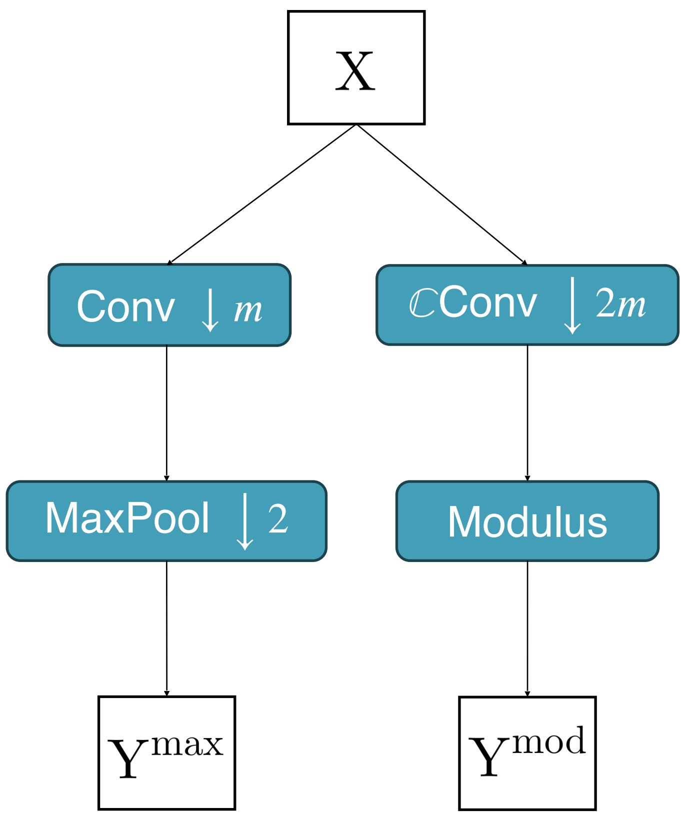
Experiments

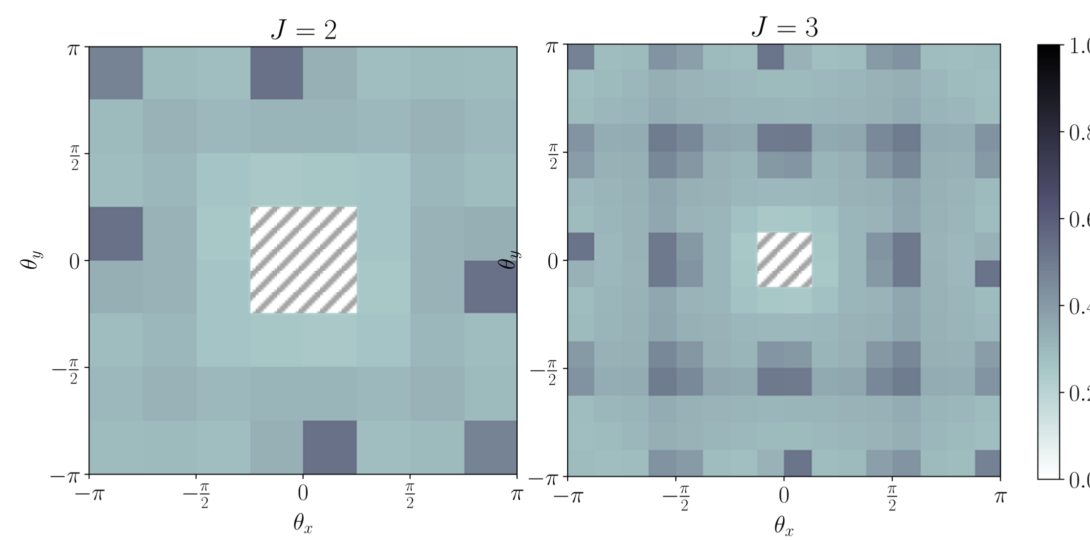
- Normalized MSE between $\mathbb{C}\mathbf{Mod}$ and $\mathbb{R}\mathbf{Max}$



Experiments

- Normalized MSE between **ℂMod** and **ℝMax**

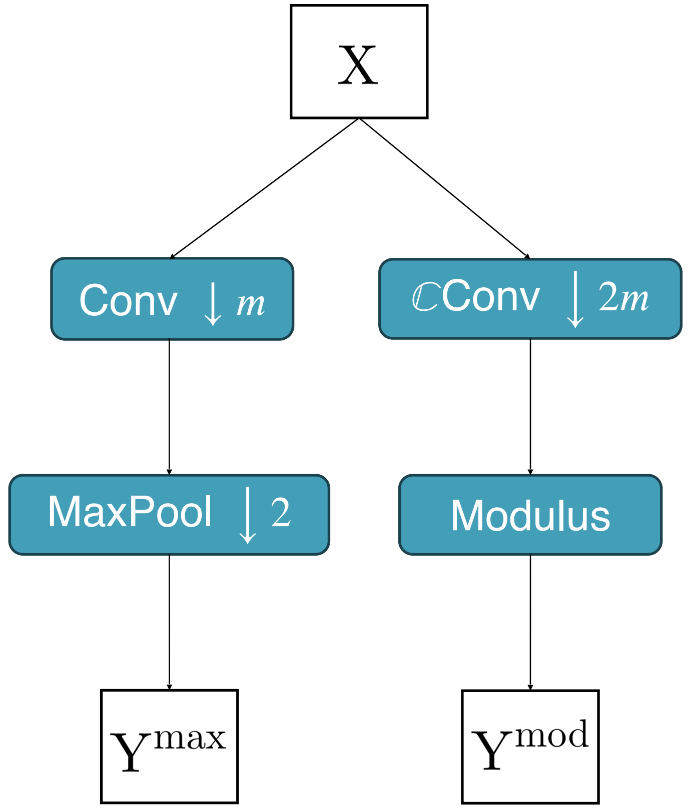


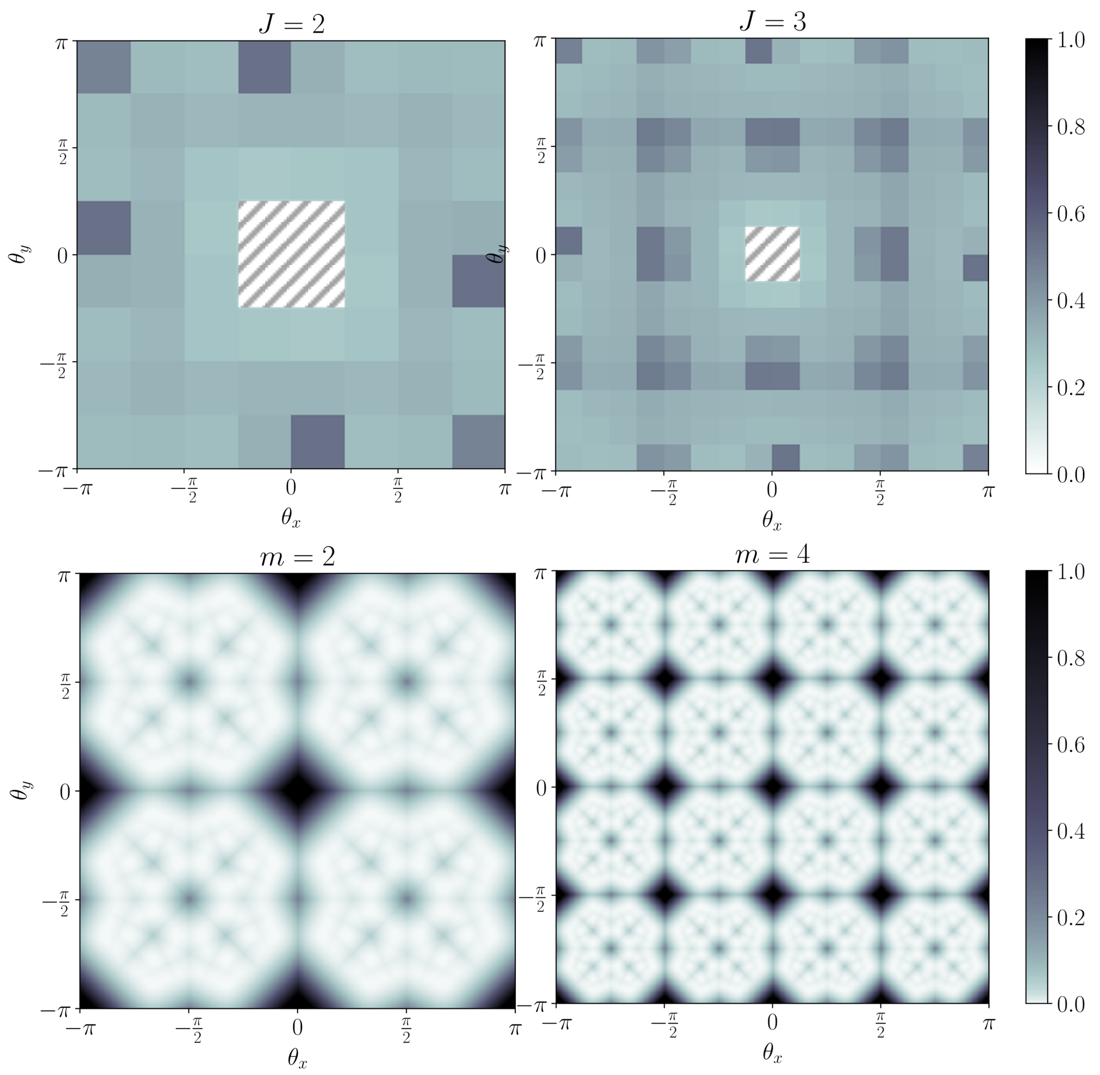


$$\rho^2 = \frac{\left\| \mathrm{Y}^{\max} - \mathrm{Y}^{\mathrm{mod}} \right\|_2^2}{\left\| \mathrm{Y}^{\mathrm{mod}} \right\|_2^2}$$

Experiments

- Normalized MSE between $\mathbb{C}$**Mod** and $\mathbb{R}$**Max**



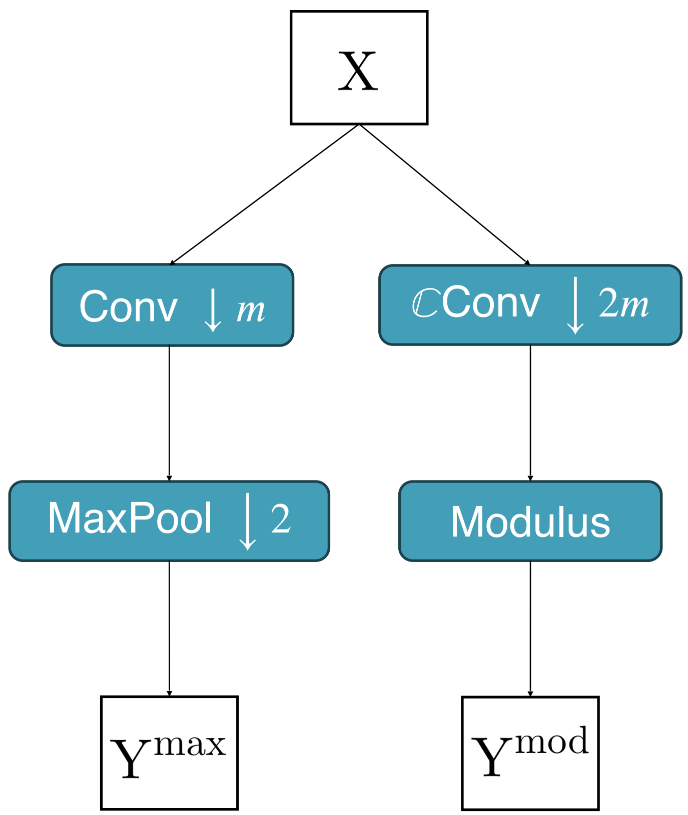


$$\rho^2 = \frac{\left\|Y^{\max} - Y^{\mathrm{mod}}\right\|_2^2}{\left\|Y^{\mathrm{mod}}\right\|_2^2}$$

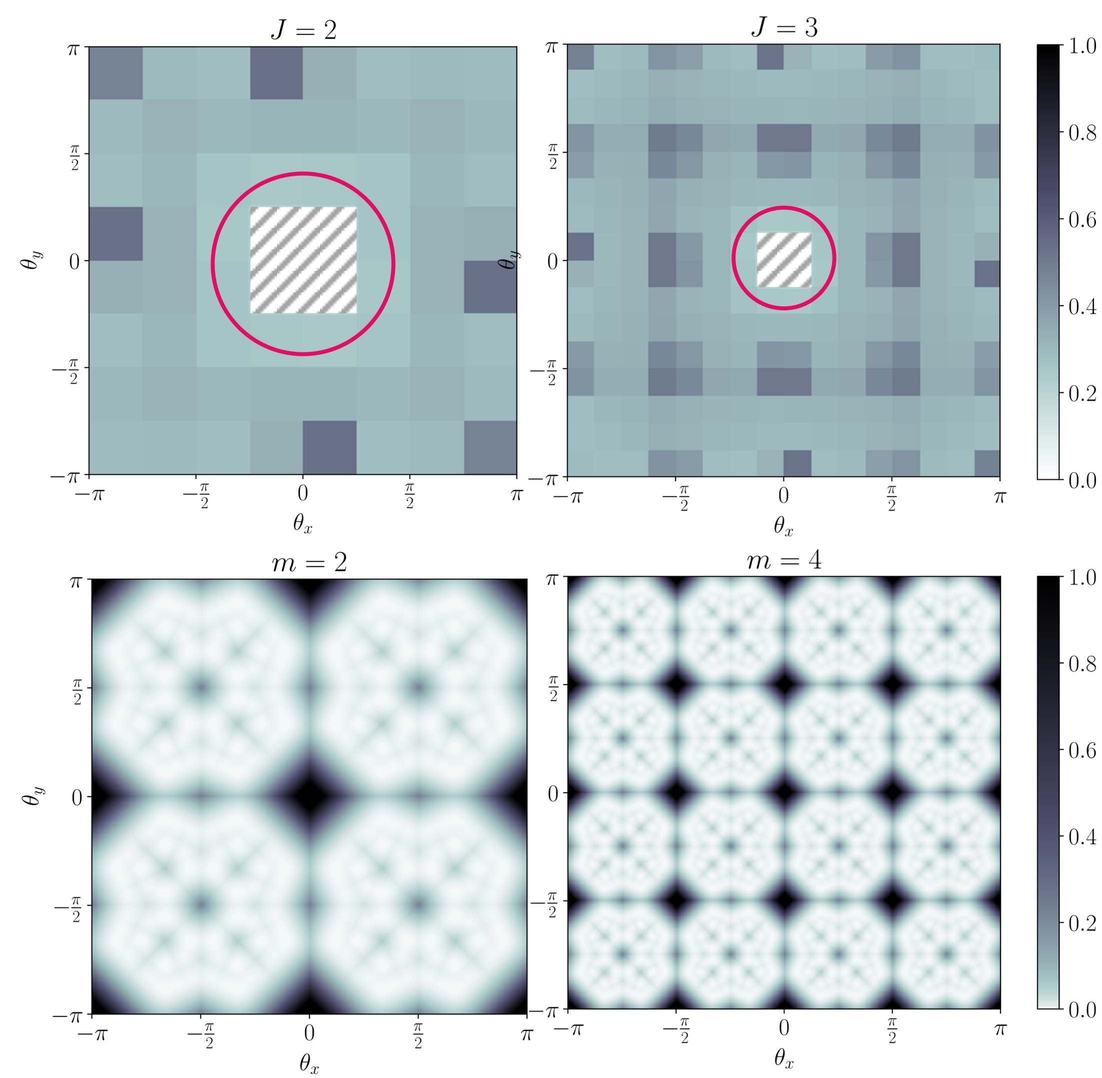
$$\boldsymbol{\theta} \mapsto \gamma_q(m\boldsymbol{\theta})^2$$

Experiments

- Normalized MSE between $\mathbb{C}$**Mod** and $\mathbb{R}$**Max**



Outside the scope of our study (low-pass filters)

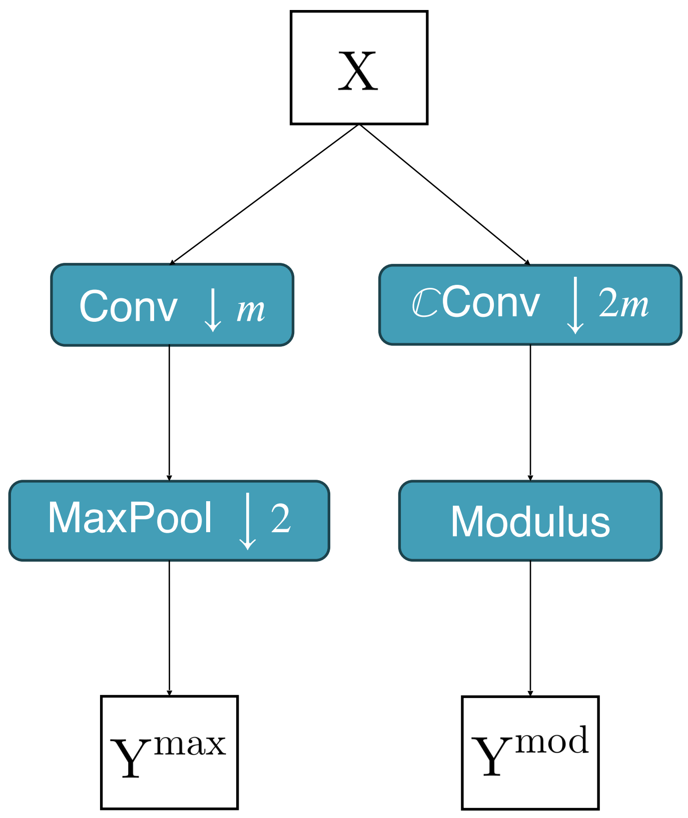


$$\rho^2 = \frac{\left\| \mathrm{Y}^{\max} - \mathrm{Y}^{\mathrm{mod}} \right\|_2^2}{\left\| \mathrm{Y}^{\mathrm{mod}} \right\|_2^2}$$

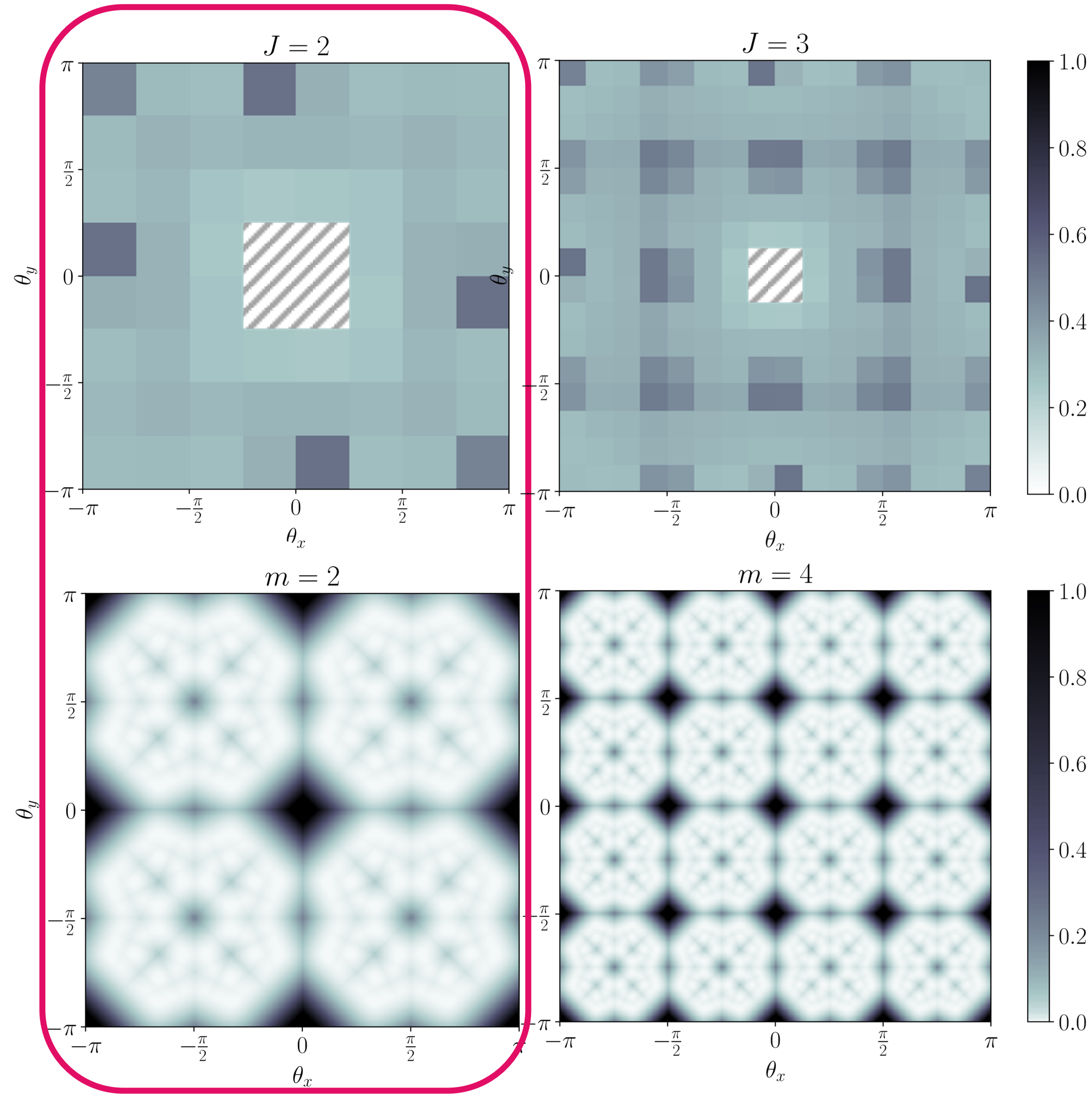
$$\boldsymbol{\theta} \mapsto \gamma_q(m\boldsymbol{\theta})^2$$

Experiments

- Normalized MSE between $\mathbb{C}\mathbf{Mod}$ and $\mathbb{R}\mathbf{Max}$



ResNet-like scenario
$J = 2,\ m = 2,\ \kappa = \pi/2$

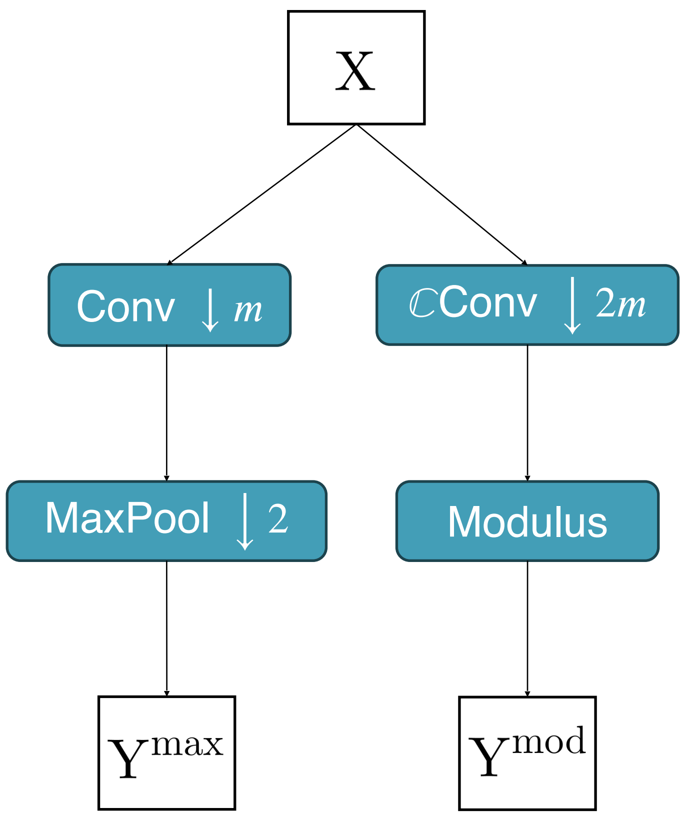


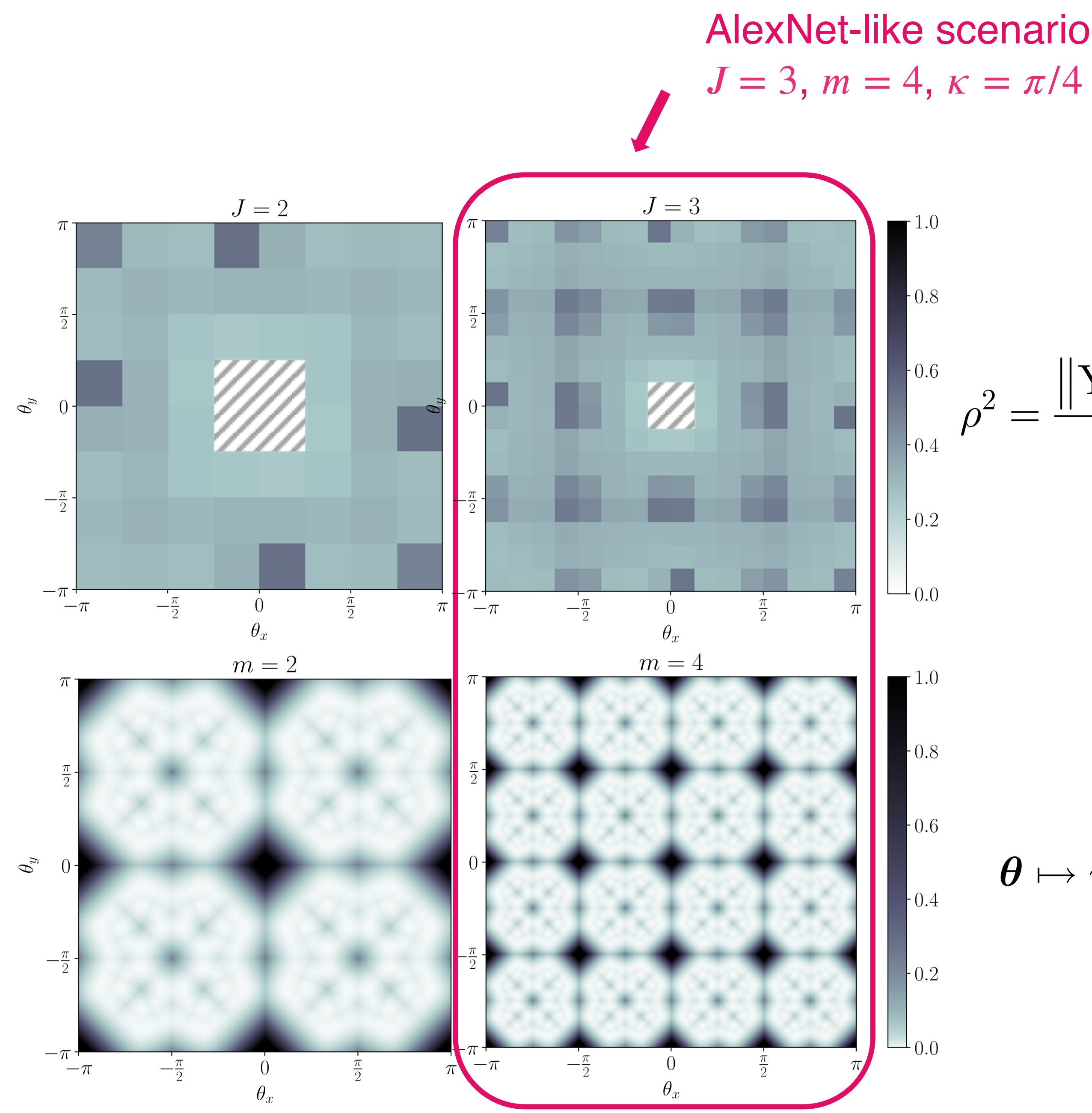
$$\rho^2 = \frac{\left\|\mathrm{Y}^{\max} - \mathrm{Y}^{\mathrm{mod}}\right\|_2^2}{\left\|\mathrm{Y}^{\mathrm{mod}}\right\|_2^2}$$

$$\boldsymbol{\theta} \mapsto \gamma_q(m\boldsymbol{\theta})^2$$

Experiments

- Normalized MSE between $\mathbb{C}$**Mod** and $\mathbb{R}$**Max**



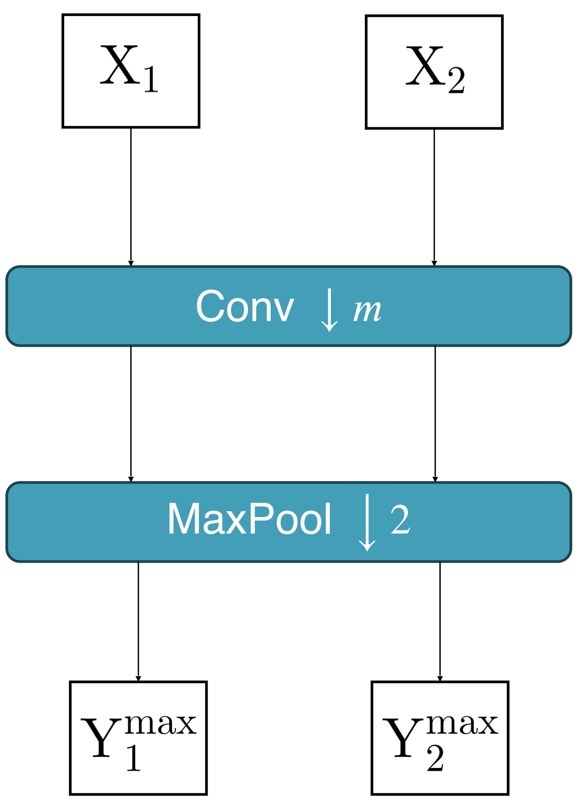


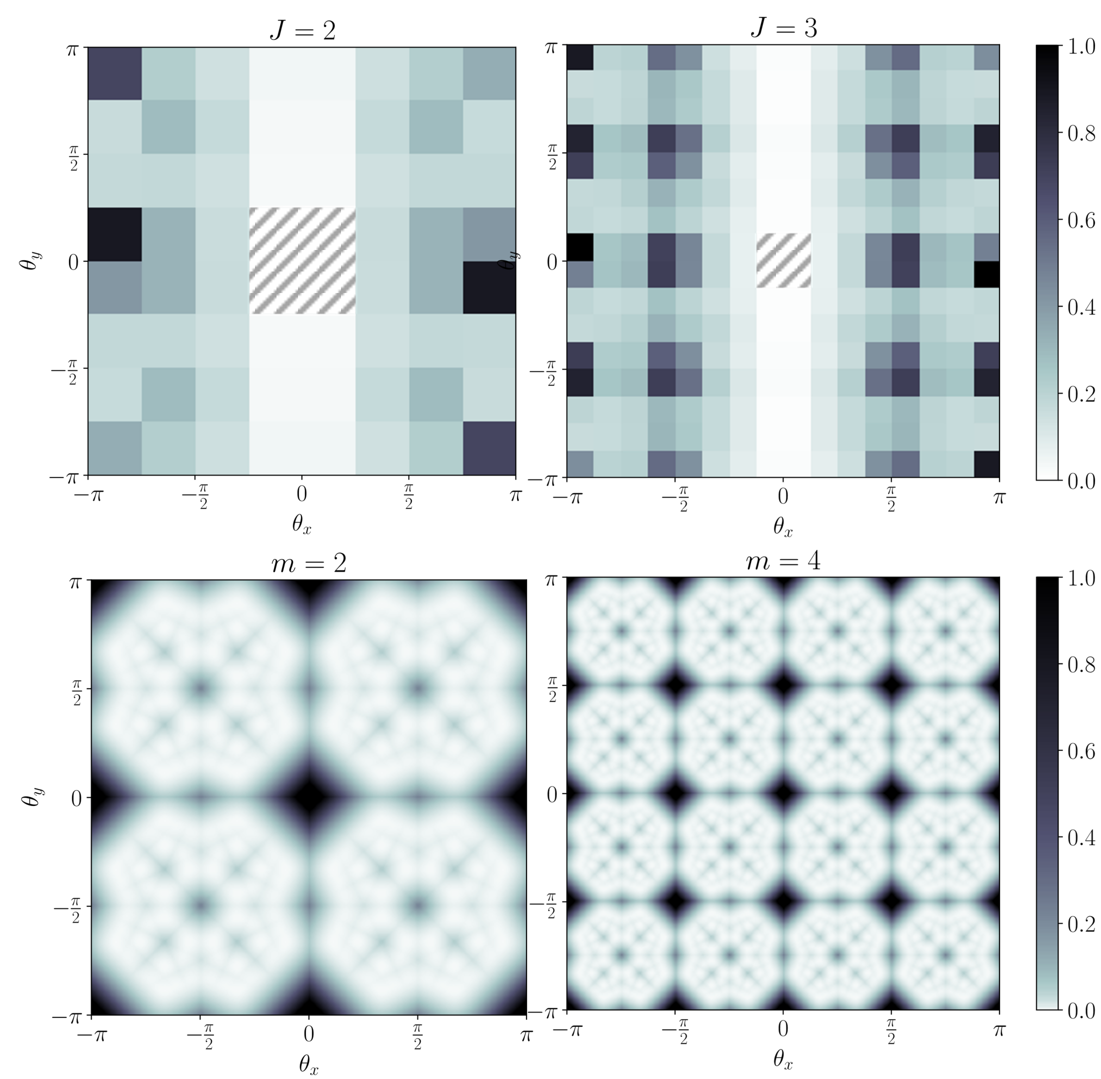
$$\rho^2 = \frac{\left\| Y^{\max} - Y^{\mathrm{mod}} \right\|_2^2}{\left\| Y^{\mathrm{mod}} \right\|_2^2}$$

$$\boldsymbol{\theta} \mapsto \gamma_q(m\boldsymbol{\theta})^2$$

Experiments

- Shift-invariance of $\mathbb{R}$**Max** outputs



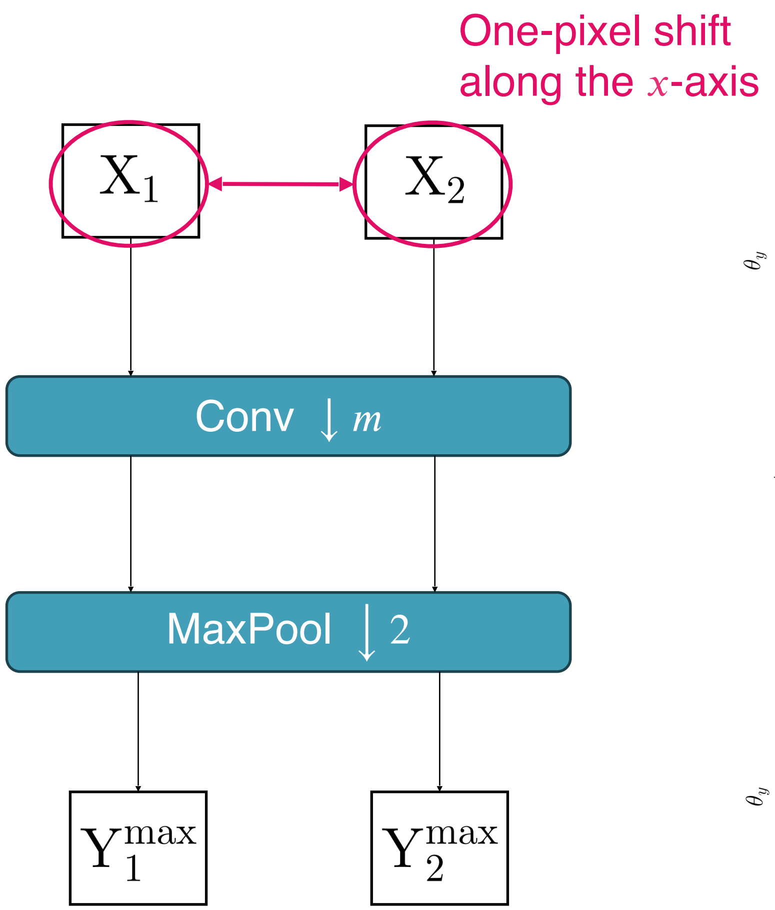


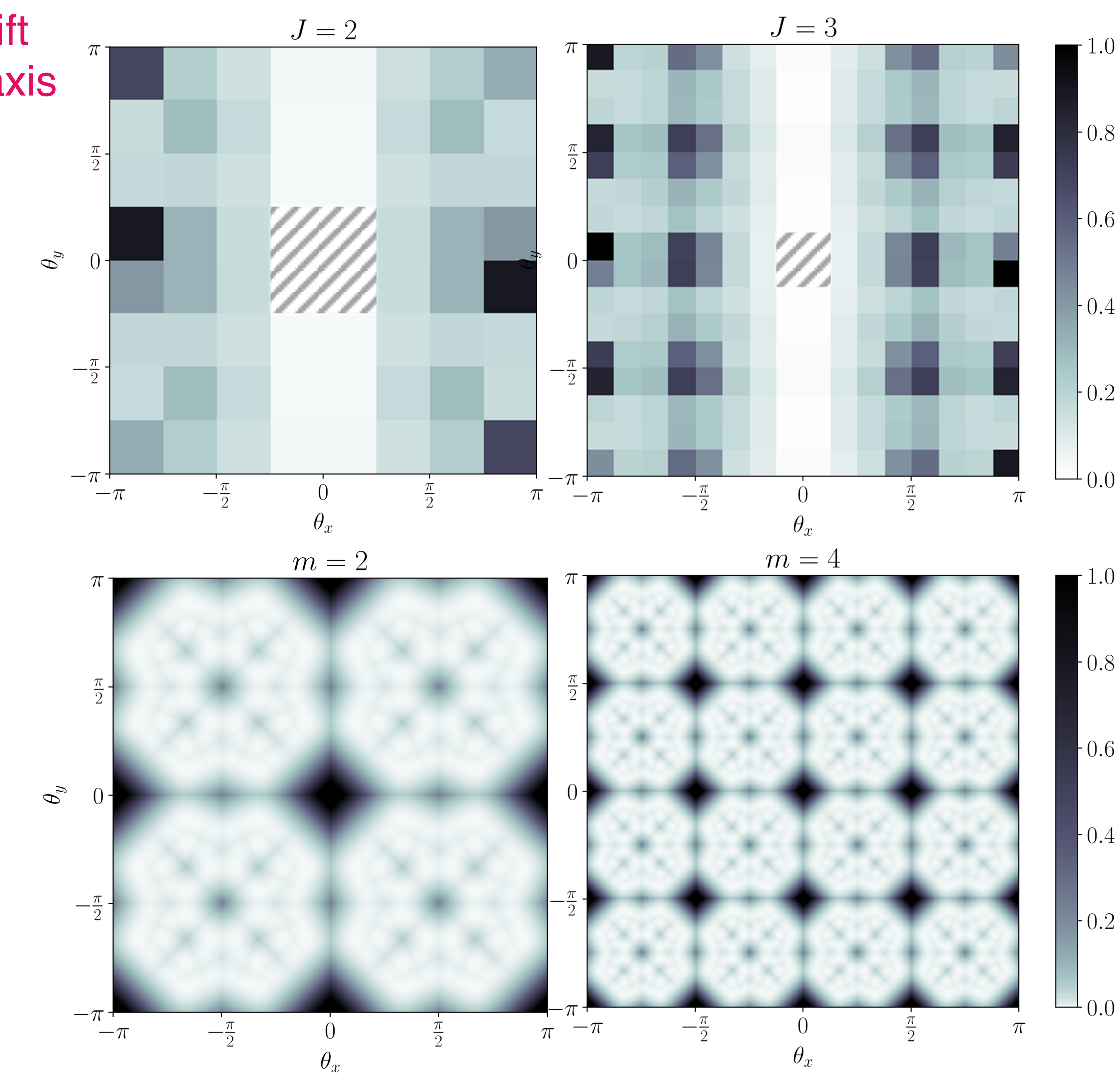
$$\rho^{\max} = \frac{\left\|Y_1^{\max} - Y_2^{\max}\right\|_2}{\left\|Y_1^{\mathrm{mod}}\right\|_2}$$

$$\boldsymbol{\theta} \mapsto \gamma_q(m\boldsymbol{\theta})^2$$

Experiments

- Shift-invariance of $\mathbb{R}$**Max** outputs





$$\rho^{\max} = \frac{\left\| Y_1^{\max} - Y_2^{\max} \right\|_2}{\left\| Y_1^{\text{mod}} \right\|_2}$$

$$\boldsymbol{\theta} \mapsto \gamma_q(m\boldsymbol{\theta})^2$$

Conclusion

Conclusion

- Exploration of the shift invariance properties captured by the max pooling operator, when applied on the first layer.

Conclusion

- Exploration of the shift invariance properties captured by the max pooling operator, when applied on the first layer.

- Bridge between complex and standard real CNNs.

Conclusion

- Exploration of the shift invariance properties captured by the max pooling operator, when applied on the first layer.
- Bridge between complex and standard real CNNs.
- Study of the discrete case in a probabilist framework.

Conclusion

- Exploration of the shift invariance properties captured by the max pooling operator, when applied on the first layer.
- Bridge between complex and standard real CNNs.
- Study of the discrete case in a probabilist framework.
- Establish a validity domain for near-shift invariance.

Conclusion

- Exploration of the shift invariance properties captured by the max pooling operator, when applied on the first layer.
- Bridge between complex and standard real CNNs.
- Study of the discrete case in a probabilist framework.
- Establish a validity domain for near-shift invariance.
- Experimental setting based on the dual-tree complex wavelet packet transform

Conclusion

- Exploration of the shift invariance properties captured by the max pooling operator, when applied on the first layer.
- Bridge between complex and standard real CNNs.
- Study of the discrete case in a probabilist framework.
- Establish a validity domain for near-shift invariance.
- Experimental setting based on the dual-tree complex wavelet packet transform
- $\mathbb{C}\mathbf{Mod}$ operator can serve as a stable proxy for $\mathbb{R}\mathbf{Max}$ enabling to improve shift invariance in CNNs architecture while preserving high-frequency information.

Publications

- Hubert Leterme, Kévin Polisano, Valérie Perrier, Karteek Alahari. **Modélisation Parcimonieuse de CNNs avec des Paquets d'Ondelettes Dual-Tree**. ORASIS 2021 - Journées francophones des jeunes chercheurs en vision par ordinateur, Centre National de la Recherche Scientifique [CNRS], Sep 2021, Saint Ferréol, France. pp.1-9. ⟨hal-03339792v2⟩

- Hubert Leterme, Kévin Polisano, Valérie Perrier, Karteek Alahari. **On the Shift Invariance of Max Pooling Feature Maps in Convolutional Neural Networks**. 2023. ⟨hal-03779434v2⟩

- Hubert Leterme, Kévin Polisano, Valérie Perrier, Karteek Alahari. **From CNNs to Shift-Invariant Twin Models Based on Complex Wavelets**. 2023. ⟨hal-03880520v2⟩

UGA Université Grenoble Alpes

Inria

LJK Laboratoire Jean Kuntzmann
Mathématiques appliquées - Informatique

PERSYVAL-Lab

MIAI Grenoble Alpes

Thank you!