# Model Fitting on the TF Plane and Noise on Synchrosqueezing Operators

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ASCETE Meeting, Grenoble

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1 Model Fitting on the TF Plane

2 Noise on Synchrosqueezing Operators

# Motivation

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- This is useful for both decomposition and denoising tasks.
- Traditional ridge-based methods suffers from resolution issues (so-called staircase effect).
- Indeed, the ridge $c(n)$ is a mapping from $\{1, \ldots, N\}$ to $\{0, \ldots, K-1\}$ because of frequency quantization.

# Linear Chirp Approximation

- Let us define

$$F_x^g(t,f) = \int_{-\infty}^{+\infty} x(u)g(u-t)e^{-i2\pi f(u-t)}du.$$

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- For a linear chirp of the form $x(t) = e^{i2\pi(at+bt^2)}$ we have

$$F_x^g(t,f) = x(t)\int_{-\infty}^{+\infty} g(u)e^{i2\pi bu^2}e^{-i2\pi(f-(a+2bt))u}du$$
$$= x(t)\widehat{g_{\phi''}}(f-\phi'(t))$$

with $g_{\phi''}(t) = g(t)e^{i2\pi\frac{\phi''}{2}t^2}$.

# Linear Chirp Approximation

- For a Gaussian window $g(t) = e^{-\sigma t^2}$, with $\sigma > 0$, we have

$$F_x^g(t,f) = x(t)\sqrt{\frac{\pi}{\sigma - i\pi\phi''}}e^{\frac{-\sigma\pi^2(f-\phi'(t))^2}{\sigma^2+\pi^2\phi''^2}}e^{\frac{-i\phi''\pi^3(f-\phi'(t))^2}{\sigma^2+\pi^2\phi''^2}}.$$

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- For a given fixed time $t = t_0$, the modulus reads

$$|F_x^g(t_0,f)| = |x(t_0)|\sqrt{\frac{\pi}{\sqrt{\sigma^2+\pi^2\phi''^2}}}\, e^{\frac{-\sigma\pi^2(f-\phi'(t_0))^2}{\sigma^2+\pi^2\phi''^2}}$$

# Traditional Ridge Estimation

- Traditional ridge estimation aims to solve

$$\max_{\mathcal{C}} \sum_{l=1}^{L} \int_{-\infty}^{+\infty} (|F_x^g(t, c_l(t))|^2 - \alpha c_l'(t)^2 - \beta c_l''(t)^2) dt,$$

with $\mathcal{C} = \{c_1(t), \ldots, c_L(t)\}$ the set of ridges and $\alpha$ and $\beta$ regularization parameters.

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- A heuristic approach simplifies the problem by solving it *time by time*. For a single mode, and setting $\alpha = \beta = 0$, we have

$$c(n\Delta t) = \text{arg}\max \int_{-\infty}^{+\infty} |F_x^g(t, f)| \delta(f - c(n\Delta t)) df,$$

  with $\delta(\cdot)$ the Dirac distribution. This makes $c(n\Delta t) \in \{0, \ldots, K-1\}$ in practice.

# Model Fitting on the TF Plane

- Based on the modulus of a linear chirp, we build the model for every $t$

$$\rho(f,\tilde{\phi}'(t),\tilde{\phi}''(t)) = \sqrt{\frac{\pi}{\sqrt{\sigma^2+\pi^2\tilde{\phi}''(t)^2}}\, e^{\frac{-\sigma\pi^2(f-\tilde{\phi}'(t))^2}{\sigma^2+\pi^2\tilde{\phi}''(t)^2}}}.$$

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- Then, we solve the following problem *looking for real parameters* $\Phi := (\tilde{\phi}', \tilde{\phi}'')_{l=1,\ldots,L}$

$$\max_{\Phi} \sum_{l=1}^{L} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} |F_x^g(t, f)| \rho(f, \tilde{\phi}_l'(t), \tilde{\phi}_l''(t)) df\, dt. \qquad (1)$$

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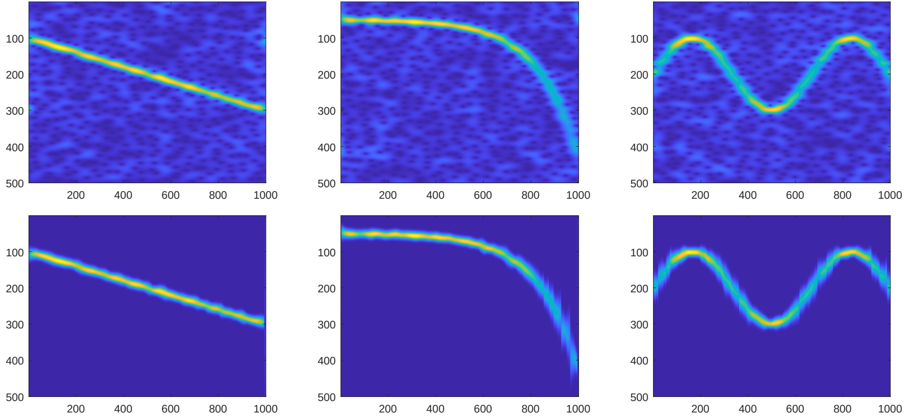
- For noiseless situation, problem (1) is convex.
- It can be easily solved without using derivatives (e.g. golden-section search).
- For noisy situations, we adopt a pre-step of hard-thresholding.
- The results proved to be robust under noisy situations.

# Examples



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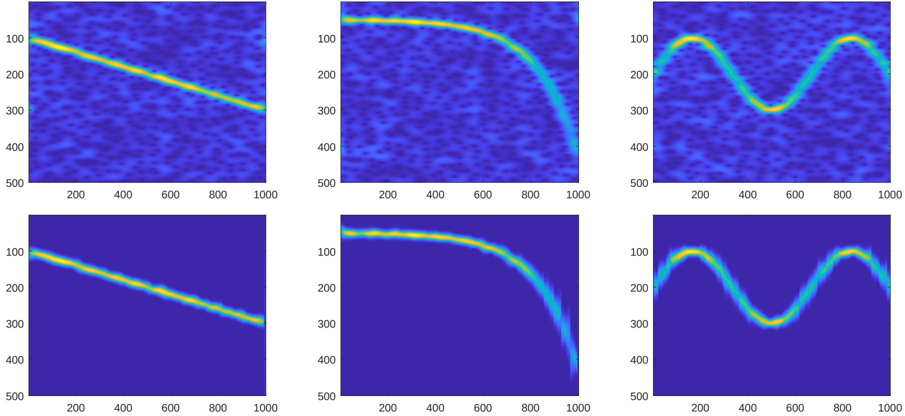
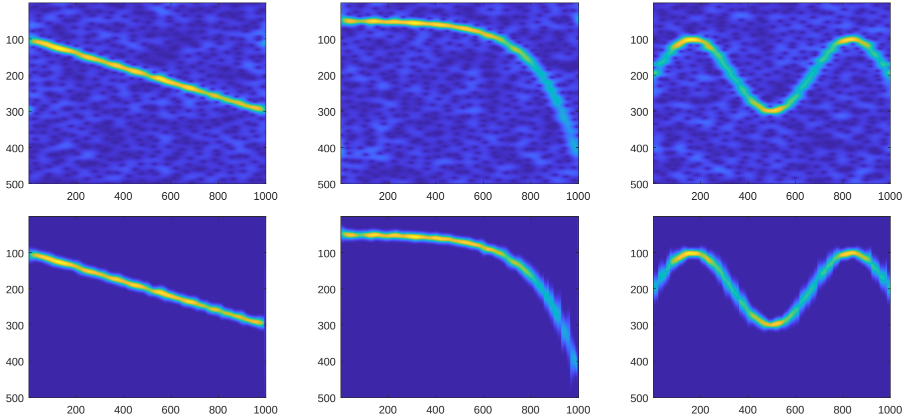



- Input SNR = 5 dB.

# Examples




- Input SNR = 5 dB.
- Reconstruction error (linear chirp): HT = 14.6211 dB; MF = 15.0425 dB.

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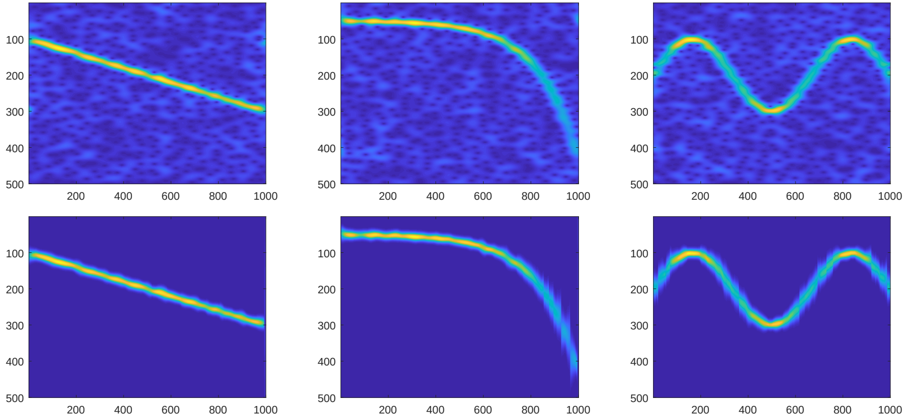


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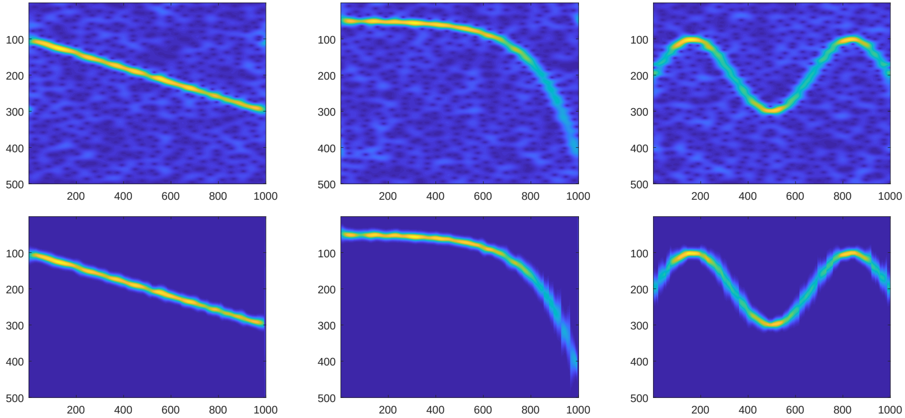


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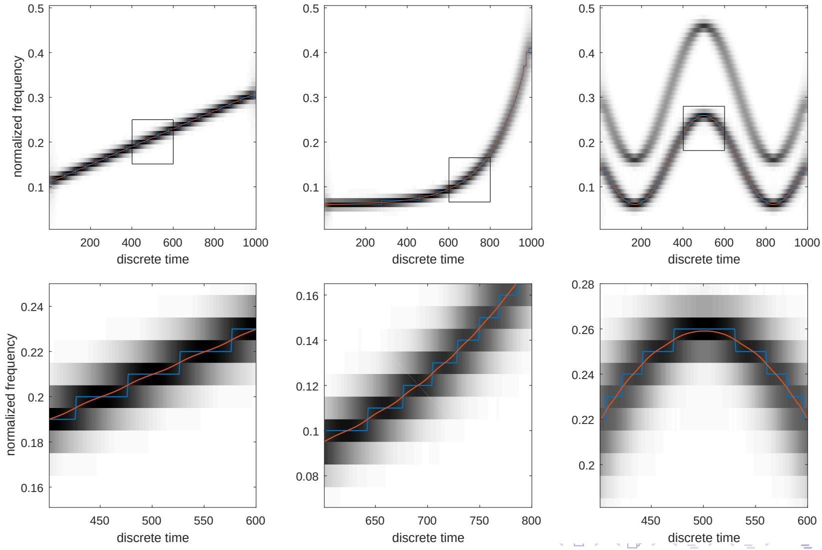
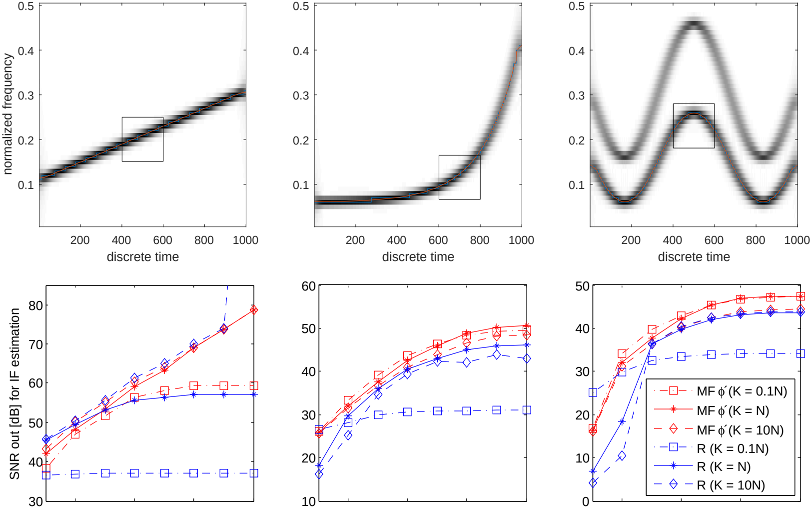
- Reconstruction error (sin. chirp): HT = 14.5288 dB; MF = 15.4312 dB.

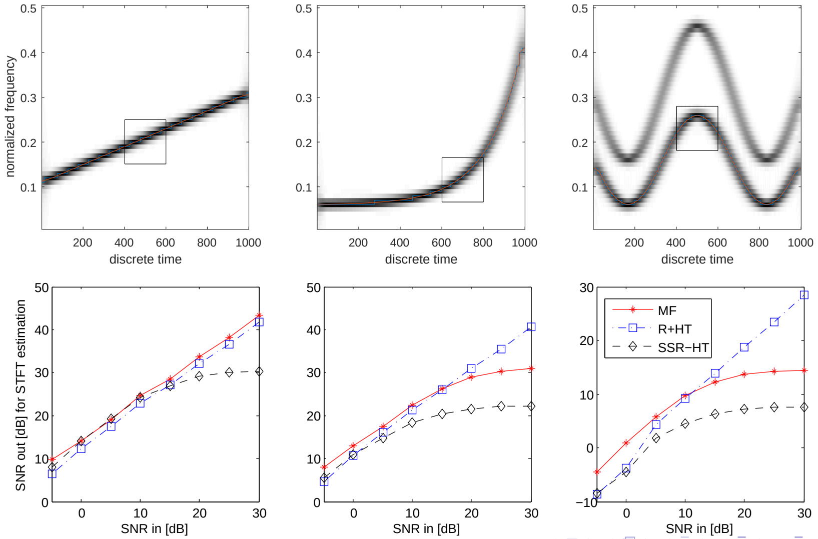
# Results


0.5
0.4
0.3
0.2
0.1
normalized frequency
200
400
600
800
1000
discrete time
0.24
0.22
0.2
0.18
0.16
450
500
550
600
650
700
750
800
0.16
0.14
0.12
0.1
0.08
0.28
0.26
0.24
0.22

# Results


normalized frequency
discrete time
SNR out [dB] for IF estimation
MF ϕ′(K = 0.1N)
MF ϕ′(K = N)
MF ϕ′(K = 10N)
R (K = 0.1N)
R (K = N)
R (K = 10N)

# Results


0.5
0.4
0.3
0.2
0.1
normalized frequency
200 400 600 800 1000
discrete time
SNR out [dB] for STFT estimation
SNR in [dB]
MF
R+HT
SSR−HT

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- Let $S_x^g(t,f) = |F_x^g(t,f)|^2$. For a noisy signal $\tilde{x} = x + n$, with $n$ zero-mean and with variance $\sigma_n^2$, we have

$$\mathbb{E}\{S_{\tilde{x}}^g(t,f)\} = S_x^g(t,f) + \mathbb{E}\{S_n^g(t,f)\}$$

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- For a fixed time $t = t_0$, we can fit the following model

$$\min_{\vec{\alpha},\vec{\beta},\vec{\gamma},\mu} \left\| S_{\tilde{x}}^g(t_0,f) - \mu - \sum_{\ell=1}^{L} \alpha_\ell e^{\frac{-2\sigma\pi^2(f-\beta_\ell)^2}{\sigma^2+\pi^2\gamma_\ell^2}} \right\|^2 \tag{2}$$

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- However, noise affects the operators in different ways.
- The impact of noise should be carefully studied.

# First Order Synchrosqueezing

- $\omega^{[1]}(t,f) = f + \Re\{\frac{1}{i2\pi}\frac{F_x^{g'}(t,f)}{F_x^g(t,f)}\}$, for $F_x^g(t,f) \neq 0$.

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- We now that $|F_n^g(t,f)| \sim \chi_2$, for $n \sim \mathcal{N}(0,\sigma_n^2)$ and we can estimate $\sigma_n$ from our data.
- So the determination of $T_1$ is possible.

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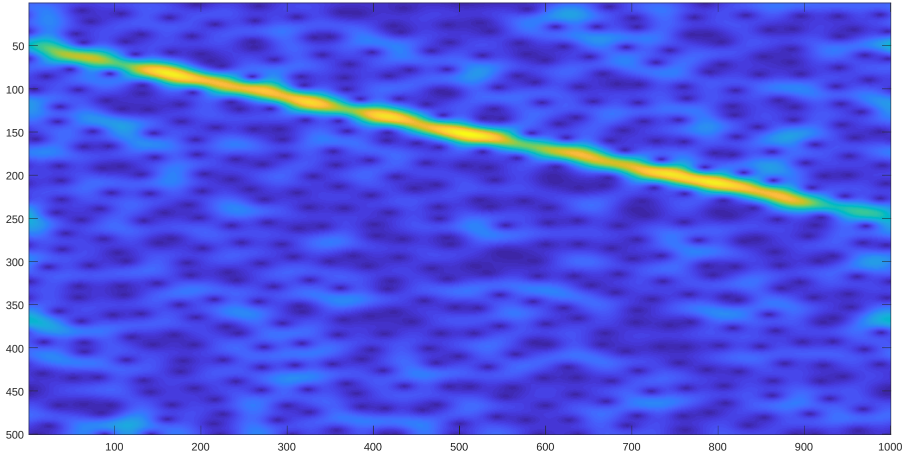

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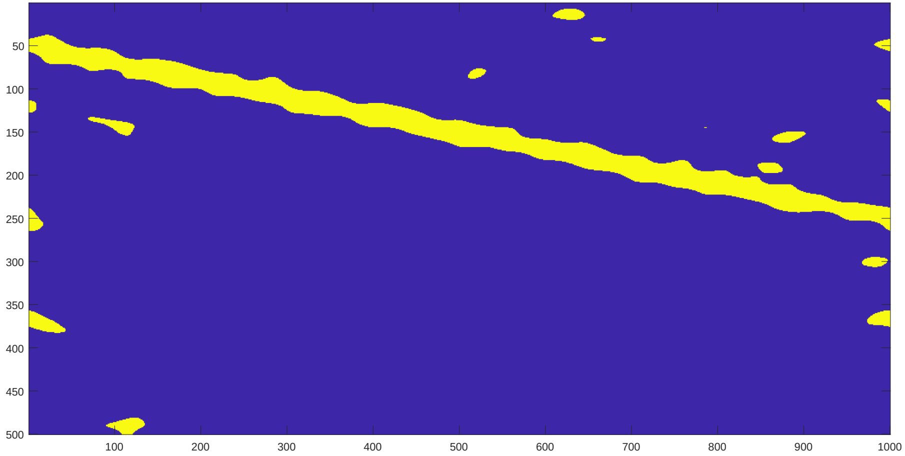

# Second Order Synchrosqueezing

- $\omega^{[2]}(t,f) = \omega^{[1]}(t,f) + \Re\{\frac{1}{i2\pi}\frac{(F_x^g)^2}{F_x^g F_x^{t^2 g} - (F_x^{tg})^2}\frac{-F_x^{tg}}{F^g}\}$, for $F_x^g F_x^{t^2 g} - (F_x^{tg})^2 \neq 0$.

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- Let us define $D = F_n^g F_n^{t^2g} - (F_n^{tg})^2$ with $n \sim \mathcal{N}(0, \sigma_n^2)$.
- We can prove that var$\{D\} = \sigma_n^2\|g\|^2\sigma_n^2\|t^2g\|^2 + 3\sigma_n^4\|tg\|^4$.

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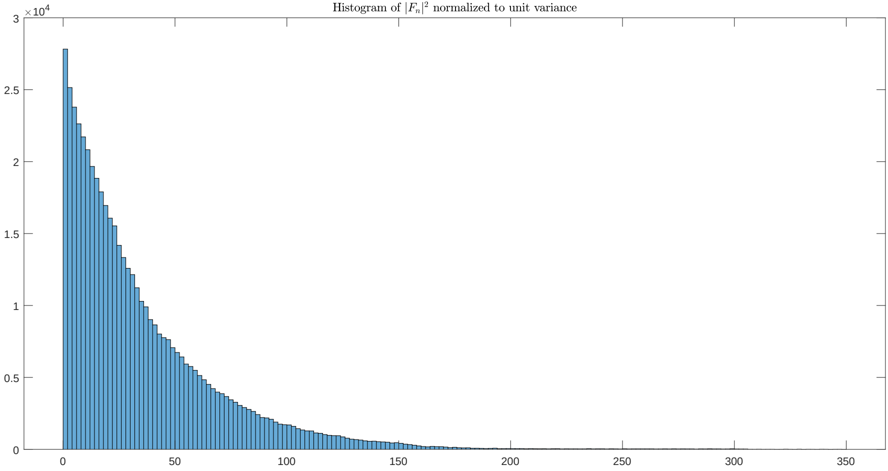

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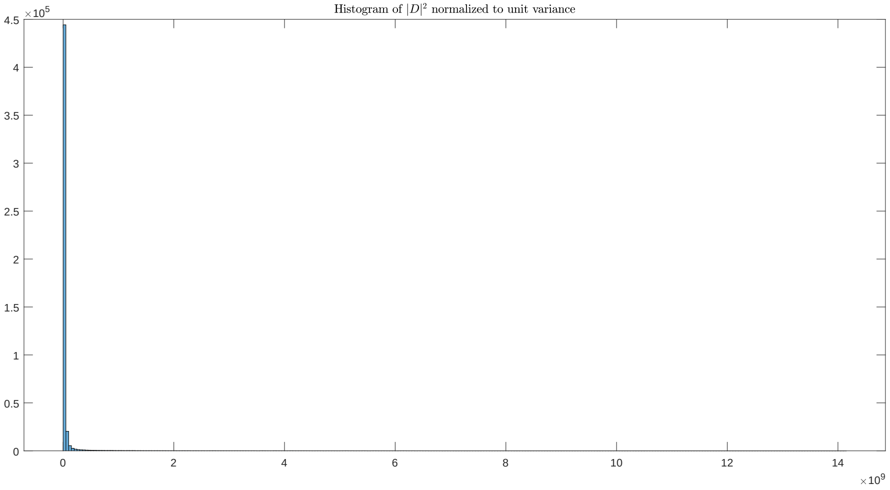

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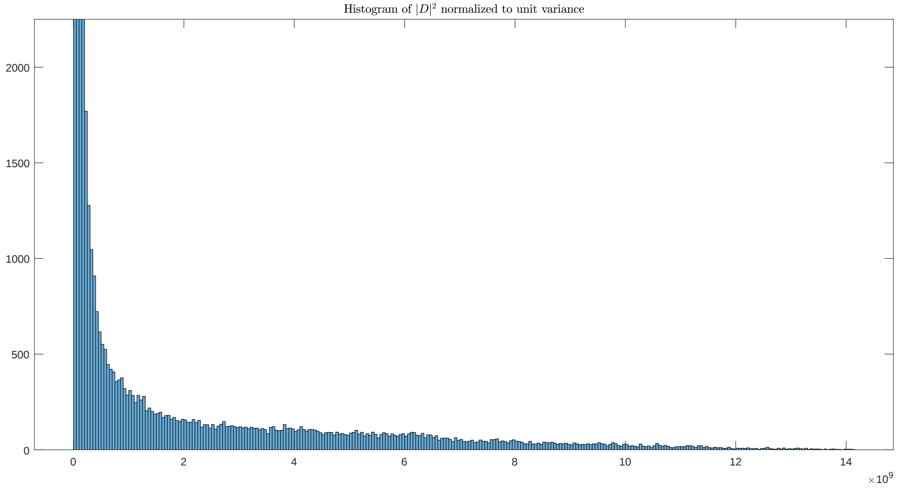

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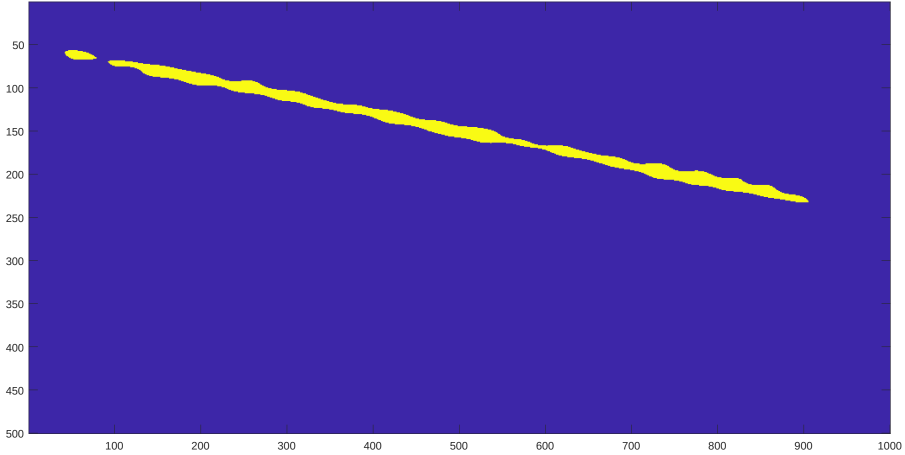

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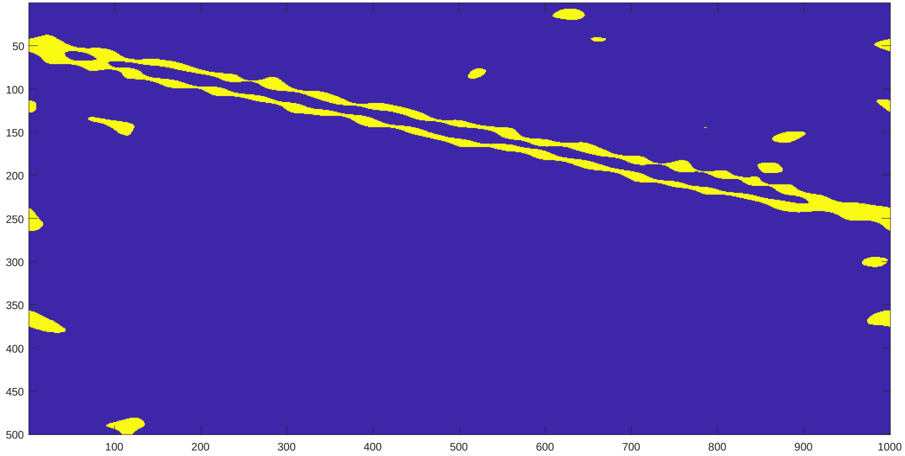

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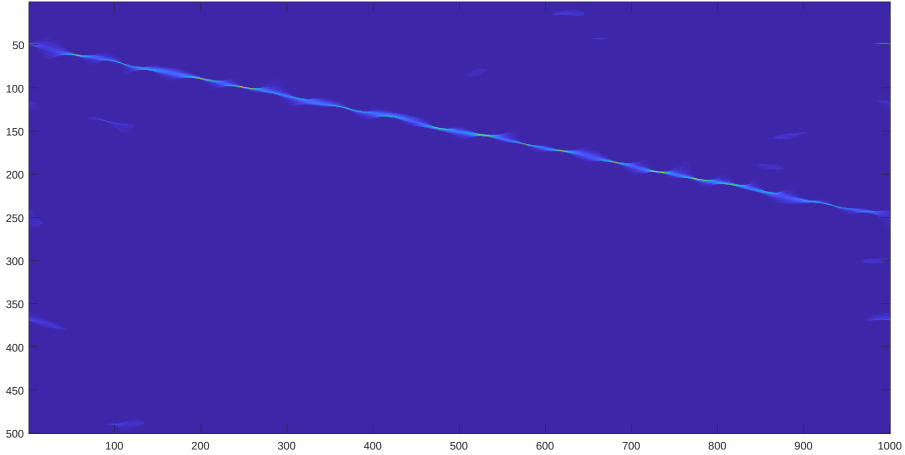

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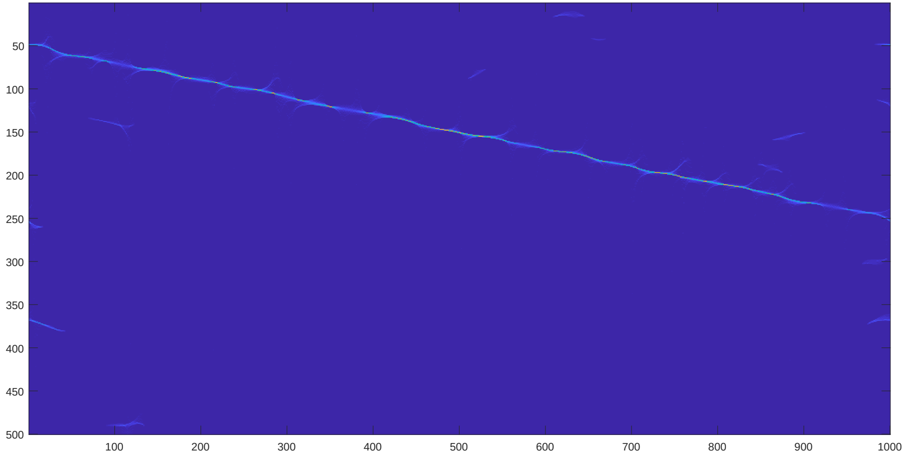

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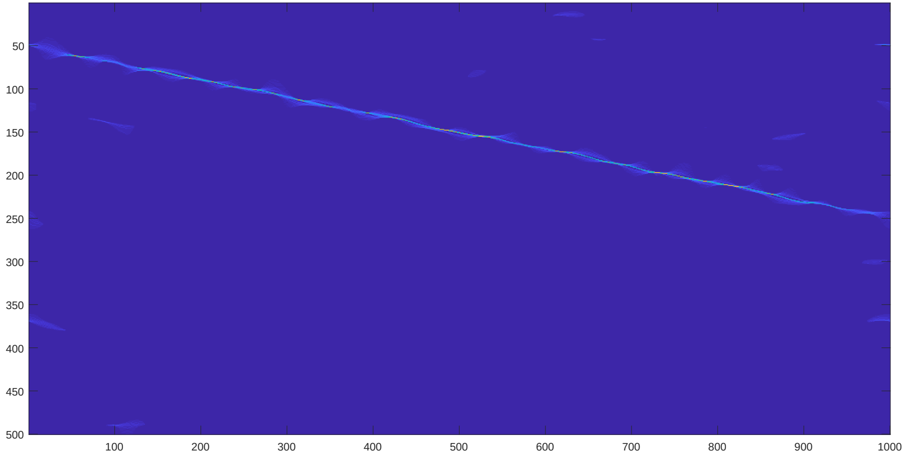

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