Further explorations on the zeros of the spectrogram

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The zeros of the spectrogram

Short-time Fourier transform (STFT) and spectrogram

For a real signal $x(t)$ its STFT will be defined as:

$$F_x(t, \omega) := \int_{-\infty}^{+\infty} x(u) g(u - t) e^{-i\omega(u - t/2)} du, \qquad (1)$$

with $g(t) = \pi^{-1/4} e^{-t^2/2}$. Then, the spectrogram is defined as:

$$S_x(t, \omega) = |F_x(t, \omega)|^2. \qquad (2)$$

STFT and Bargmann transform

Considering $z = \omega + it$, then the STFT can be written as:

$$F_x(t, \omega) = \mathcal{F}_x(z) \exp\left(-|z|^2/2\right), \qquad (3)$$

where $\mathcal{F}_x(z)$ is the *Bargmann transform*.

The zeros of the spectrogram

The Haddamard-Weierstrass Factorization

Being $\mathcal{F}_x(z)$ an entire function of order 2, it admits the following factorization:

$$\mathcal{F}_x(z) = z^m e^{Q(z)} \prod_n \left(1 - \frac{z}{z_n}\right) \exp\left(\frac{z}{z_n} + \frac{z^2}{2z_n^2}\right), \qquad (4)$$

where z_n are the zeros of $\mathcal{F}_x(z)$, m is the order of a (possible) zero at the origin, and $Q(z)$ is a quadratic polynomial.

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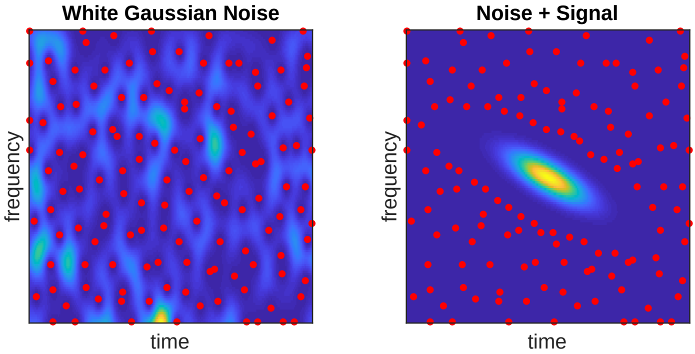
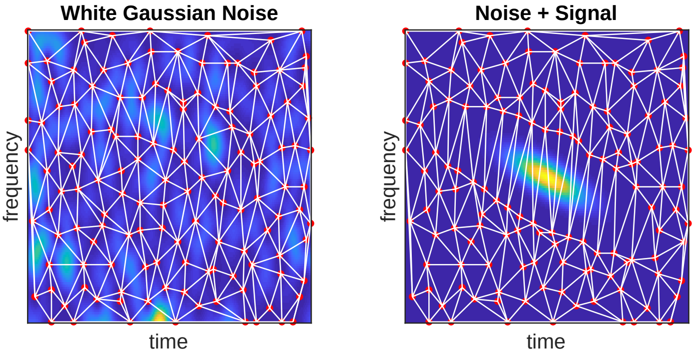
- This fact can be harnessed for signal detection (Flandrin 2015).
- Zeros also corresponds to the zeros of a *Gaussian Analytic Function* (Bardenet 2018).


White Gaussian Noise
frequency
time
Noise + Signal
frequency
time


White Gaussian Noise
frequency
time
Noise + Signal
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Some work in progress

1. Spatial statistics on the zeros of the spectrogram on-the-fly.
2. Benchmark of controlled applications: zeros-based methods vs. ridges/large values based methods.
3. A noise assisted approach for signal domain detection.

1. Detecting zeros on-the-fly

Objective

- To do spatial statistics (envelope tests, triangulation) time slice after time slice.

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- Detection of zeros of $S(t, \omega)$ using time slices is more challenging than in the time-frequency plane.
- Zeros of the analytic function $\mathcal{F}_x(z)$ corresponds to the local minima of $|\mathcal{F}_x(z)|$ (Escudero 2021).

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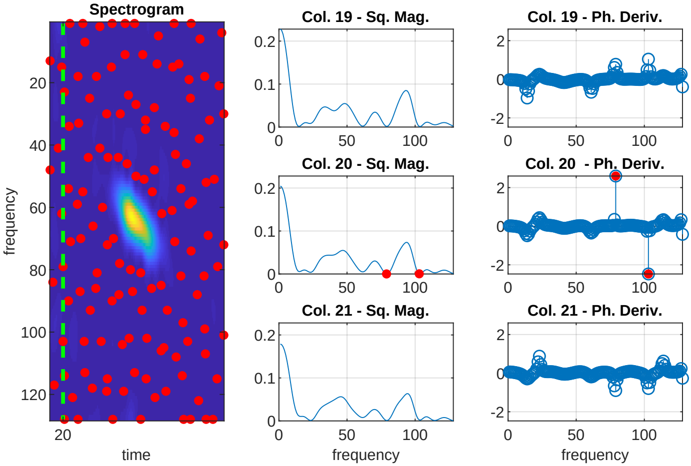
- Zeros of the analytic function $\mathcal{F}_x(z)$ corresponds to the local minima of $|\mathcal{F}_x(z)|$ (Escudero 2021).
- The derivative of the phase of the STFT, however, exhibits a pole-behaviour near the zeros (Auger 2012, Balazs 2015).

1. Detecting zeros on-the-fly



2. Benchmark of controlled applications

Objective

- To determine what kind of regimes make the zeros based methods a better option than traditional (for instance, ridge-based) methods.

- Study a number of tasks (i.e. detection, filtering) and types of signals.
- Explore the impact of different criteria for determining the signal domain.

3. A noise assisted approach for signal domain detection

Objective

- To obtain useful information from the position of zeros when adding different realizations of noise to a signal.

- Inspired by other noise assisted methods (Wu & Huang 2009, Colominas 2012).
- The zeros of the original signal suffer small position changes when some low amplitude noise is added.
- Zeros nearer the signal should be more restricted in movement because they are repelled from the signal domain.

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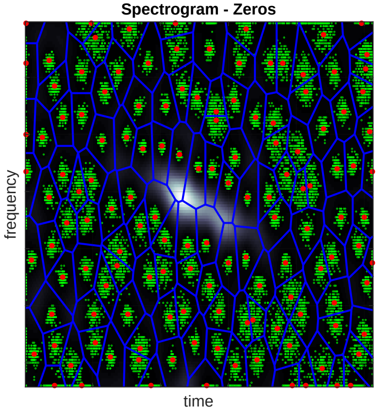
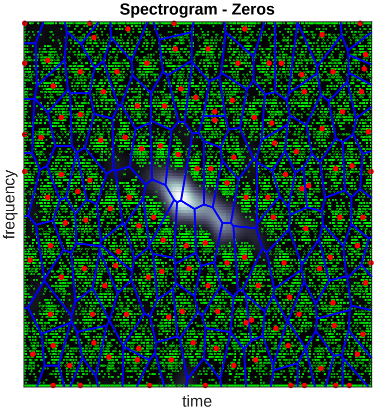
- Zeros nearer the signal should be more restricted in movement because they are repelled from the signal domain.
- An *optimal* level of noise might exists in order to retrieve information from the changes in the distribution of zeros.

3. A noise assisted approach for signal domain detection

Noise Realizations' SNR: 10 dB



Noise Realizations' SNR: 0 dB



- Red dots are the original zeros of the signal.
- Green dots are the new zeros created by different realizations.

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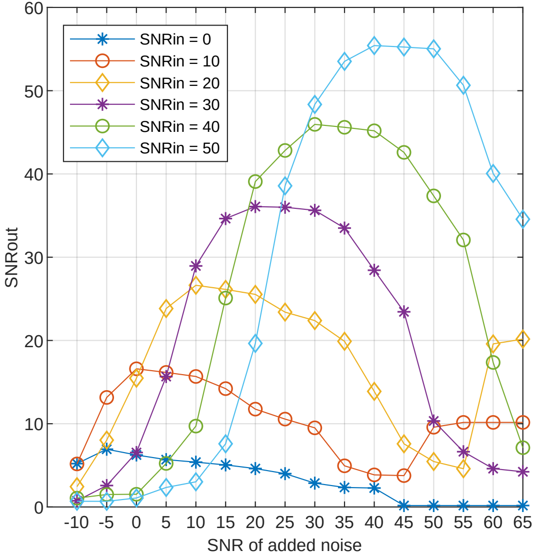

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